Higher

Mathematics

Trigonometry

Contents

CfE Edition

5 Compound Angles EF

When we add or subtract angles, the result is a **compound angle**.

For example, $45° + 30°$ is a compound angle. Using a calculator, we find:

- $\sin(45° + 30°) = \sin(75°) = 0{\cdot}966$
- $\sin(45°) + \sin(30°) = 1{\cdot}207$ (both to 3 d.p.).

This shows that $\sin(A + B)$ is *not* equal to $\sin A + \sin B$. Instead, we can use the following identities:

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$
$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$
$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$
$$\cos(A - B) = \cos A \cos B + \sin A \sin B.$$

These are given in the exam in a condensed form:

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B.$$

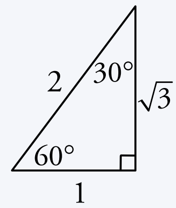
EXAMPLES

1. Expand and simplify $\cos(x° + 60°)$.

$$\cos(x° + 60°) = \cos x° \cos 60° - \sin x° \sin 60°$$
$$= \tfrac{1}{2}\cos x° - \tfrac{\sqrt{3}}{2}\sin x°.$$

Remember

The exact value triangle:



2. Show that $\sin(a + b) = \sin a \cos b + \cos a \sin b$ for $a = \frac{\pi}{6}$ and $b = \frac{\pi}{3}$.

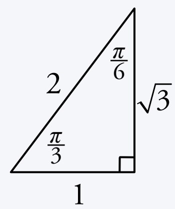
$$\begin{aligned}\text{LHS} &= \sin(a + b) \\ &= \sin\left(\tfrac{\pi}{6} + \tfrac{\pi}{3}\right) \\ &= \sin\left(\tfrac{\pi}{2}\right) \\ &= 1.\end{aligned}$$

$$\begin{aligned}\text{RHS} &= \sin a \cos b + \cos a \sin b \\ &= \sin\tfrac{\pi}{6}\cos\tfrac{\pi}{3} + \cos\tfrac{\pi}{6}\sin\tfrac{\pi}{3} \\ &= \left(\tfrac{1}{2} \times \tfrac{1}{2}\right) + \left(\tfrac{\sqrt{3}}{2} \times \tfrac{\sqrt{3}}{2}\right) \\ &= \tfrac{1}{4} + \tfrac{3}{4} = 1.\end{aligned}$$

Since $\text{LHS} = \text{RHS}$, the claim is true for $a = \frac{\pi}{6}$ and $b = \frac{\pi}{3}$.

Remember

The exact value triangle:

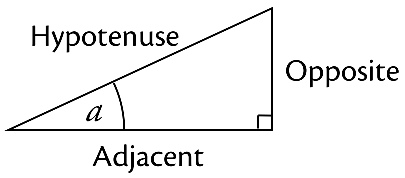


3. Find the exact value of $\sin 75°$.

$$\begin{aligned}\sin 75° &= \sin(45° + 30°)\\ &= \sin 45° \cos 30° + \cos 45° \sin 30°\\ &= \left(\tfrac{1}{\sqrt{2}} \times \tfrac{\sqrt{3}}{2}\right) + \left(\tfrac{1}{\sqrt{2}} \times \tfrac{1}{2}\right)\\ &= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}\\ &= \frac{\sqrt{3}+1}{2\sqrt{2}}\\ &= \frac{\sqrt{6}+\sqrt{2}}{4}.\end{aligned}$$

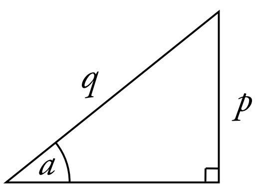
Finding Trigonometric Ratios

You should already be familiar with the following formulae (SOH CAH TOA).



$$\sin a = \frac{\text{Opposite}}{\text{Hypotenuse}} \qquad \cos a = \frac{\text{Adjacent}}{\text{Hypotenuse}} \qquad \tan a = \frac{\text{Opposite}}{\text{Adjacent}}.$$

If we have $\sin a = \frac{p}{q}$ where $0 < a < \frac{\pi}{2}$, then we can form a right-angled triangle to represent this ratio.



Since $\sin a = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{p}{q}$ then:

- the side opposite a has length p;
- the hypotenuse has length q.

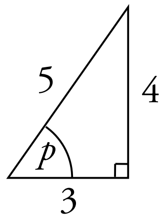
The length of the unknown side can be found using Pythagoras's Theorem.

Once the length of each side is known, we can find $\cos a$ and $\tan a$ using SOH CAH TOA.

The method is similar if we know $\cos a$ and want to find $\sin a$ or $\tan a$.

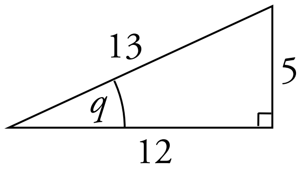
EXAMPLES

4. Acute angles p and q are such that $\sin p = \frac{4}{5}$ and $\sin q = \frac{5}{13}$. Show that $\sin(p+q) = \frac{63}{65}$.



$\sin p = \frac{4}{5}$

$\cos p = \frac{3}{5}$



$\sin q = \frac{5}{13}$

$\cos q = \frac{12}{13}$

$$\begin{aligned}\sin(p+q) &= \sin p \cos q + \cos p \sin q \\ &= \left(\tfrac{4}{5} \times \tfrac{12}{13}\right) + \left(\tfrac{3}{5} \times \tfrac{5}{13}\right) \\ &= \tfrac{48}{65} + \tfrac{15}{65} \\ &= \tfrac{63}{65}.\end{aligned}$$

Note
Since "Show that" is used in the question, all of this working is required.

Using compound angle formulae to confirm identities

EXAMPLES

5. Show that $\sin\left(x - \frac{\pi}{2}\right) = -\cos x$.

$$\begin{aligned}&\sin\left(x - \tfrac{\pi}{2}\right) \\ &= \sin x \cos \tfrac{\pi}{2} - \cos x \sin \tfrac{\pi}{2} \\ &= \sin x \times 0 - \cos x \times 1 \\ &= -\cos x.\end{aligned}$$

6. Show that $\dfrac{\sin(s+t)}{\cos s \cos t} = \tan s + \tan t$ for $\cos s \neq 0$ and $\cos t \neq 0$.

$$\begin{aligned}\frac{\sin(s+t)}{\cos s \cos t} &= \frac{\sin s \cos t + \cos s \sin t}{\cos s \cos t} \\ &= \frac{\sin s \cos t}{\cos s \cos t} + \frac{\cos s \sin t}{\cos s \cos t} \\ &= \frac{\sin s}{\cos s} + \frac{\sin t}{\cos t} \\ &= \tan s + \tan t.\end{aligned}$$

Remember
$\dfrac{\sin x}{\cos x} = \tan x.$

6 Double-Angle Formulae

EF

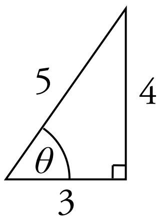
Using the compound angle identities with $A = B$, we obtain expressions for $\sin 2A$ and $\cos 2A$. These are called **double-angle formulae**.

$$\sin 2A = 2\sin A \cos A$$
$$\cos 2A = \cos^2 A - \sin^2 A$$
$$= 2\cos^2 A - 1$$
$$= 1 - 2\sin^2 A.$$

Note that these are given in the exam.

EXAMPLES

1. Given that $\tan\theta = \frac{4}{3}$, where $0 < \theta < \frac{\pi}{2}$, find the exact value of $\sin 2\theta$ and $\cos 2\theta$.



$\sin\theta = \frac{4}{3}$

$\cos\theta = \frac{3}{5}$

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta \\ &= 2\times\tfrac{4}{5}\times\tfrac{3}{5} \\ &= \tfrac{24}{25}.\end{aligned}$$

$$\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= \left(\tfrac{3}{5}\right)^2 - \left(\tfrac{4}{5}\right)^2 \\ &= \tfrac{9}{25} - \tfrac{16}{25} \\ &= -\tfrac{7}{25}.\end{aligned}$$

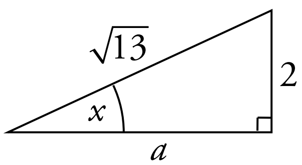
Note
Any of the $\cos 2A$ formulae could have been used here.

2. Given that $\cos 2x = \frac{5}{13}$, where $0 < x < \pi$, find the exact values of $\sin x$ and $\cos x$.

Since $\cos 2x = 1 - 2\sin^2 x$,

$$\begin{aligned}1 - 2\sin^2 x &= \tfrac{5}{13} \\ 2\sin^2 x &= \tfrac{8}{13} \\ \sin^2 x &= \tfrac{8}{26} \\ &= \tfrac{4}{13} \\ \sin x &= \pm\tfrac{2}{\sqrt{13}}.\end{aligned}$$

We are told that $0 < x < \pi$, so only $\sin x = \frac{2}{\sqrt{13}}$ is possible.



$$a = \sqrt{\sqrt{13}^2 - 2^2} = \sqrt{13 - 4} = \sqrt{9} = 3.$$

So $\cos x = \frac{3}{\sqrt{13}}$.

7 Further Trigonometric Equations RC

We will now consider trigonometric equations where double-angle formulae can be used to find solutions. These equations will involve:

- $\sin 2x$ and either $\sin x$ or $\cos x$;
- $\cos 2x$ and $\cos x$;
- $\cos 2x$ and $\sin x$.

Remember
The double-angle formulae are given in the exam.

Solving equations involving sin2*x* and either sin*x* or cos*x*

EXAMPLE

1. Solve $\sin 2x° = -\sin x°$ for $0 \leq x < 360$.

$$2\sin x° \cos x° = -\sin x°$$
$$2\sin x° \cos x° + \sin x° = 0$$
$$\sin x° (2\cos x° + 1) = 0$$

- Replace $\sin 2x$ using the double angle formula
- Take all terms to one side, making the equation equal to zero
- Factorise the expression and solve

$\sin x° = 0$

$x = 0$ or 180 or $\cancel{360}$

$2\cos x° + 1 = 0$

$\cos x° = -\frac{1}{2}$

$x = 180 - 60$ or $180 + 60$

$= 120$ or 240.

✓ S | A
✓ T | C

$x = \cos^{-1}\left(\frac{1}{2}\right)$

$= 60$.

So $x = 0$ or 120 or 180 or 240.

Solving equations involving cos2*x* and cos*x*

EXAMPLE

2. Solve $\cos 2x = \cos x$ for $0 \leq x \leq 2\pi$.

$$\cos 2x = \cos x$$
$$2\cos^2 x - 1 = \cos x$$
$$2\cos^2 x - \cos x - 1 = 0$$
$$(2\cos x + 1)(\cos x - 1) = 0$$

- Replace $\cos 2x$ by $2\cos^2 x - 1$
- Take all terms to one side, making a quadratic equation in $\cos x$
- Solve the quadratic equation (using factorisation or the quadratic formula)

$2\cos x + 1 = 0$

$\cos x = -\frac{1}{2}$

$x = \pi - \frac{\pi}{3}$ or $\pi + \frac{\pi}{3}$

$= \frac{2\pi}{3}$ or $\frac{4\pi}{3}$

✓ S | A
✓ T | C

$x = \cos^{-1}\left(\frac{1}{2}\right)$

$= \frac{\pi}{3}$

$\cos x - 1 = 0$

$\cos x = 1$

$x = 0$ or 2π.

So $x = 0$ or $\frac{2\pi}{3}$ or $\frac{4\pi}{3}$ or 2π.

Solving equations involving cos2*x* and sin*x*

EXAMPLE

3. Solve $\cos 2x = \sin x$ for $0 < x < 2\pi$.

$$\begin{aligned} \cos 2x &= \sin x \\ 1 - 2\sin^2 x &= \sin x \\ 2\sin^2 x + \sin x - 1 &= 0 \\ (2\sin x - 1)(\sin x + 1) &= 0 \end{aligned}$$

- Replace $\cos 2x$ by $1 - 2\sin^2 x$
- Take all terms to one side, making a quadratic equation in $\sin x$
- Solve the quadratic equation (using factorisation or the quadratic formula)

$$\begin{aligned} 2\sin x - 1 &= 0 \\ \sin x &= \tfrac{1}{2} \\ x &= \tfrac{\pi}{6} \quad \text{or} \quad \pi - \tfrac{\pi}{6} \\ &= \tfrac{\pi}{6} \quad \text{or} \quad \tfrac{5\pi}{6} \end{aligned}$$

✓ S	A ✓
T	C

$$\begin{aligned} x &= \sin^{-1}\left(\tfrac{1}{2}\right) \\ &= \tfrac{\pi}{6} \end{aligned}$$

$$\begin{aligned} \sin x + 1 &= 0 \\ \sin x &= -1 \\ x &= \tfrac{3\pi}{2}. \end{aligned}$$

So $x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$ or $\frac{3\pi}{2}$.