Higher Mathematics

Quadratics

Contents

CfE Edition

Polynomials and Quadratics

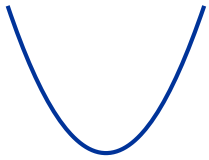
1 Quadratics EF

A **quadratic** has the form $ax^2 + bx + c$ where a, b, and c are any real numbers, provided $a \neq 0$.

You should already be familiar with the following.

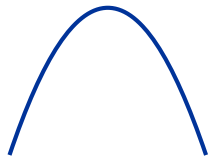
The graph of a quadratic is called a **parabola**. There are two possible shapes:

concave up (if $a > 0$)



This has a minimum turning point

concave down (if $a < 0$)



This has a maximum turning point

To find the roots (i.e. solutions) of the quadratic equation $ax^2 + bx + c = 0$, we can use:

- factorisation;
- completing the square (see Section 3);
- the quadratic formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ (this is *not* given in the exam).

EXAMPLES

1. Find the roots of $x^2 - 2x - 3 = 0$.

$$\begin{aligned} x^2 - 2x - 3 &= 0 \\ (x+1)(x-3) &= 0 \\ x + 1 = 0 \quad &\text{or} \quad x - 3 = 0 \\ x = -1 \quad & \quad\quad x = 3. \end{aligned}$$

2. Solve $x^2 + 8x + 16 = 0$.

$$\begin{aligned} x^2 + 8x + 16 &= 0 \\ (x+4)(x+4) &= 0 \\ x + 4 = 0 \quad &\text{or} \quad x + 4 = 0 \\ x = -4 \quad & \quad\quad x = -4. \end{aligned}$$

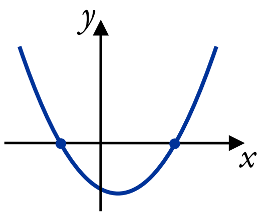
3. Find the roots of $x^2 + 4x - 1 = 0$.

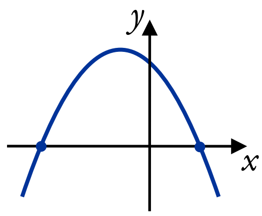
We cannot factorise $x^2 + 4x - 1$, but we can use the quadratic formula:

$$
\begin{aligned}
x &= \frac{-4 \pm \sqrt{4^2 - 4 \times 1 \times (-1)}}{2 \times 1} \\
&= \frac{-4 \pm \sqrt{16 + 4}}{2} \\
&= \frac{-4 \pm \sqrt{20}}{2} \\
&= -\tfrac{4}{2} \pm \tfrac{\sqrt{4}\sqrt{5}}{2} \\
&= -2 \pm \sqrt{5}.
\end{aligned}
$$

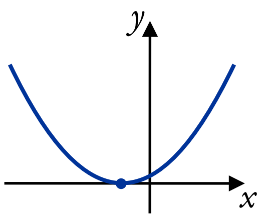
Note

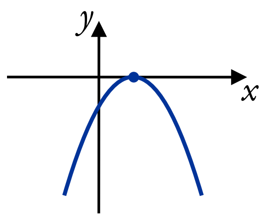
- If there are two distinct solutions, the curve intersects the x-axis twice.



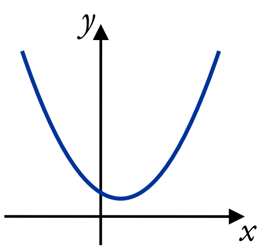


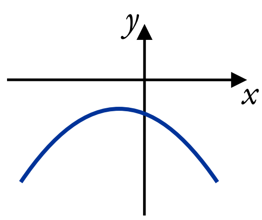
- If there is one repeated solution, the turning point lies on the x-axis.





- If $b^2 - 4ac < 0$ when using the quadratic formula, there are no points where the curve intersects the x-axis.





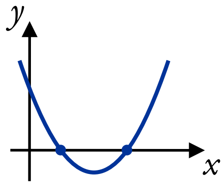
2 The Discriminant

EF

Given $ax^2 + bx + c$, we call $b^2 - 4ac$ the **discriminant**.

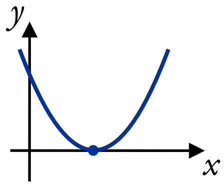
This is the part of the quadratic formula which determines the number of real roots of the equation $ax^2 + bx + c = 0$.

- If $b^2 - 4ac > 0$, the roots are real and unequal (distinct).



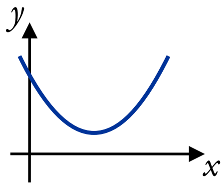
two roots

- If $b^2 - 4ac = 0$, the roots are real and equal (i.e. a repeated root).



one root

- If $b^2 - 4ac < 0$, the roots are not real; the parabola does not cross the x-axis.



no real roots

EXAMPLE

1. Find the nature of the roots of $9x^2 + 24x + 16 = 0$.

$a = 9$ $\quad b^2 - 4ac = 24^2 - 4 \times 9 \times 16$

$b = 24$ $\quad = 576 - 576$

$c = 16$ $\quad = 0$

Since $b^2 - 4ac = 0$, the roots are real and equal.

2. Find the values of q such that $6x^2 + 12x + q = 0$ has real roots.

Since $6x^2 + 12x + q = 0$ has real roots, $b^2 - 4ac \geq 0$:

$a = 6$ $\quad b^2 - 4ac \geq 0$

$b = 12$ $\quad 12^2 - 4 \times 6 \times q \geq 0$

$c = q$ $\quad 144 - 24q \geq 0$

$$144 \geq 24q$$

$$24q \leq 144$$

$$q \leq 6.$$

3. Find the range of values of k for which the equation $kx^2 + 2x - 7 = 0$ has no real roots.

For no real roots, we need $b^2 - 4ac < 0$:

$a = k$

$b = 2$

$c = -7$

$$\begin{aligned} b^2 - 4ac &< 0 \\ 2^2 - 4 \times k \times (-7) &< 0 \\ 4 + 28k &< 0 \\ 28k &< -4 \\ k &< -\frac{4}{28} \\ k &< -\tfrac{1}{7}. \end{aligned}$$

4. Show that $(2k+4)x^2 + (3k+2)x + (k-2) = 0$ has real roots for all real values of k.

$a = 2k + 4$

$b = 3k + 2$

$c = k - 2$

$$\begin{aligned} & b^2 - 4ac \\ &= (3k+2)^2 - 4(2k+4)(k-2) \\ &= 9k^2 + 12k + 4 - (2k+4)(4k-8) \\ &= 9k^2 + 12k + 4 - 8k^2 + 32 \\ &= k^2 + 12k + 36 \\ &= (k+6)^2. \end{aligned}$$

Since $b^2 - 4ac = (k+6)^2 \geq 0$, the roots are always real.

3 Completing the Square EF

The process of writing $y = ax^2 + bx + c$ in the form $y = a(x+p)^2 + q$ is called **completing the square**.

Once in "completed square" form we can determine the turning point of any parabola, including those with no real roots.

The axis of symmetry is $x = -p$ and the turning point is $(-p, q)$.

The process relies on the fact that $(x+p)^2 = x^2 + 2px + p^2$. For example, we can write the expression $x^2 + 4x$ using the bracket $(x+2)^2$ since when multiplied out this gives the terms we want – with an extra constant term.

This means we can rewrite the expression $x^2 + kx$ using $\left(x + \frac{k}{2}\right)^2$ since this gives us the correct x^2 and x terms, with an extra constant.

We will use this to help complete the square for $y = 3x^2 + 12x - 3$.

Step 1
Make sure the equation is in the form $y = ax^2 + bx + c$.

$$y = 3x^2 + 12x - 3.$$

Step 2
Take out the x^2-coefficient as a factor of the x^2 and x terms.

$$y = 3\left(x^2 + 4x\right) - 3.$$

Step 3
Replace the $x^2 + kx$ expression and compensate for the extra constant.

$$\begin{aligned} y &= 3\left((x+2)^2 - 4\right) - 3 \\ &= 3(x+2)^2 - 12 - 3. \end{aligned}$$

Step 4
Collect together the constant terms.

$$y = 3(x+2)^2 - 15.$$

Now that we have completed the square, we can see that the parabola with equation $y = 3x^2 + 12x - 3$ has turning point $(-2, -15)$.

EXAMPLES

1. Write $y = x^2 + 6x - 5$ in the form $y = (x + p)^2 + q$.

$$\begin{aligned} y &= x^2 + 6x - 5 \\ &= (x+3)^2 - 9 - 5 \\ &= (x+3)^2 - 14. \end{aligned}$$

Note
You can always check your answer by expanding the brackets.

2. Write $x^2 + 3x - 4$ in the form $(x + p)^2 + q$.

$$\begin{aligned} & x^2 + 3x - 4 \\ &= \left(x + \tfrac{3}{2}\right)^2 - \tfrac{9}{4} - 4 \\ &= \left(x + \tfrac{3}{2}\right)^2 - \tfrac{25}{4}. \end{aligned}$$

3. Write $y = x^2 + 8x - 3$ in the form $y = (x+a)^2 + b$ and then state:
   (i) the axis of symmetry, and
   (ii) the minimum turning point of the parabola with this equation.

$$\begin{aligned} y &= x^2 + 8x - 3 \\ &= (x+4)^2 - 16 - 3 \\ &= (x+4)^2 - 19. \end{aligned}$$

(i) The axis of symmetry is $x = -4$.

(ii) The minimum turning point is $(-4, -19)$.

4. A parabola has equation $y = 4x^2 - 12x + 7$.
   (a) Express the equation in the form $y = (x+a)^2 + b$.
   (b) State the turning point of the parabola and its nature.

(a)
$$\begin{aligned} y &= 4x^2 - 12x + 7 \\ &= 4\left(x^2 - 3x\right) + 7 \\ &= 4\left(\left(x - \tfrac{3}{2}\right)^2 - \tfrac{9}{4}\right) + 7 \\ &= 4\left(x - \tfrac{3}{2}\right)^2 - 9 + 7 \\ &= 4\left(x - \tfrac{3}{2}\right)^2 - 2. \end{aligned}$$

(b) The turning point is $\left(\tfrac{3}{2}, -2\right)$ and is a minimum.

Remember
If the coefficient of x^2 is positive then the parabola is concave up.

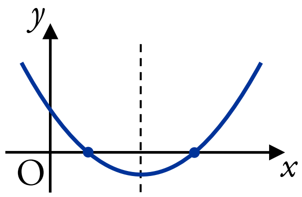
4 Sketching Parabolas EF

The method used to sketch the curve with equation $y = ax^2 + bx + c$ depends on how many times the curve intersects the *x*-axis.

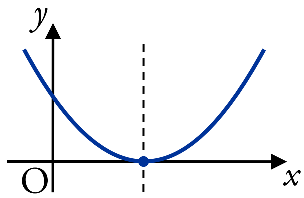
We have met curve sketching before. However, when sketching parabolas, we *do not* need to use calculus. We know there is only one turning point, and we have methods for finding it.

Parabolas with one or two roots

- Find the *x*-axis intercepts by factorising or using the quadratic formula.
- Find the *y*-axis intercept (i.e. where $x = 0$).
- The turning point is on the axis of symmetry:



The axis of symmetry is halfway between two distinct roots.



A repeated root lies on the axis of symmetry.

Parabolas with no real roots

- There are no *x*-axis intercepts.
- Find the *y*-axis intercept (i.e. where $x = 0$).
- Find the turning point by completing the square.

EXAMPLES

1. Sketch the graph of $y = x^2 - 8x + 7$.

Since $b^2 - 4ac = (-8)^2 - 4 \times 1 \times 7 > 0$, the parabola crosses the *x*-axis twice.

The *y*-axis intercept $(x = 0)$:

$$y = (0)^2 - 8(0) + 7$$
$$= 7$$
$$(0, 7).$$

The *x*-axis intercepts $(y = 0)$:

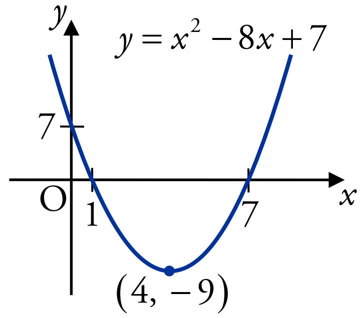
$$x^2 - 8x + 7 = 0$$
$$(x - 1)(x - 7) = 0$$
$$x - 1 = 0 \quad \text{or} \quad x - 7 = 0$$
$$x = 1 \qquad\qquad x = 7$$
$$(1, 0) \qquad\qquad (7, 0).$$

The axis of symmetry lies halfway between $x = 1$ and $x = 7$, i.e. $x = 4$, so the *x*-coordinate of the turning point is 4.

We can now find the y-coordinate:

$$y = (4)^2 - 8(4) + 7$$
$$= 16 - 32 + 7$$
$$= -9.$$

So the turning point is $(4, -9)$.



2. Sketch the parabola with equation $y = -x^2 - 6x - 9$.

Since $b^2 - 4ac = (-6)^2 - 4 \times (-1) \times (-9) = 0$, there is a repeated root.

The y-axis intercept $(x = 0)$:

$$y = -(0)^2 - 6(0) - 9$$
$$= -9$$

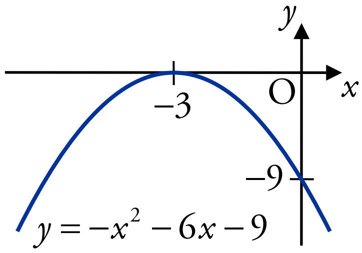
$(0, -9)$.

The x-axis intercept $(y = 0)$:

$$-x^2 - 6x - 9 = 0$$
$$-\left(x^2 + 6x + 9\right) = 0$$
$$(x+3)(x+3) = 0$$
$$x + 3 = 0$$
$$x = -3$$

$(-3, 0)$.

Since there is a repeated root, $(-3, 0)$ is the turning point.



3. Sketch the curve with equation $y = 2x^2 - 8x + 13$.

Since $b^2 - 4ac = (-8)^2 - 4 \times 2 \times 13 < 0$, there are no real roots.

The y-axis intercept $(x = 0)$:

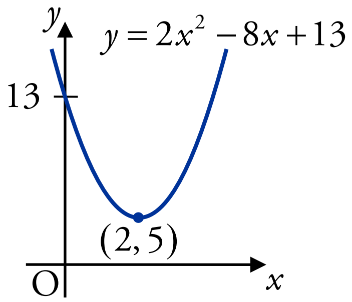
$$y = 2(0)^2 - 8(0) + 13$$
$$= 13$$

$(0, 13)$.

Complete the square:

$$y = 2x^2 - 8x + 13$$
$$= 2\left(x^2 - 4x\right) + 13$$
$$= 2(x-2)^2 - 8 + 13$$
$$= 2(x-2)^2 + 5.$$

So the turning point is $(2, 5)$.



5 Determining the Equation of a Parabola RC

Given the equation of a parabola, we have seen how to sketch its graph. We will now consider the opposite problem: finding an equation for a parabola based on information about its graph.

We can find the equation given:

- the roots and another point,
- the turning point and another point.

When we know the roots

If a parabola has roots $x = a$ and $x = b$ then its equation is of the form

$$y = k(x-a)(x-b)$$

where k is some constant.

If we know another point on the parabola, then we can find the value of k.

EXAMPLES

1. A parabola passes through the points $(1,0)$, $(5,0)$ and $(0,3)$.

   Find the equation of the parabola.

Since the parabola cuts the x-axis where $x = 1$ and $x = 5$, the equation is of the form:

$$y = k(x-1)(x-5).$$

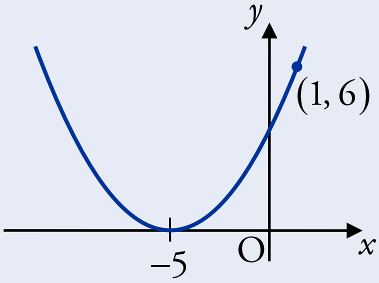
To find k, we use the point $(0,3)$:

$$y = k(x-1)(x-5)$$
$$3 = k(0-1)(0-5)$$
$$3 = 5k$$
$$k = \tfrac{3}{5}.$$

So the equation of the parabola is:

$$y = \tfrac{3}{5}(x-1)(x-5)$$
$$= \tfrac{3}{5}\left(x^2 - 6x + 5\right)$$
$$= \tfrac{3}{5}x^2 - \tfrac{18}{5}x + 3.$$

2. Find the equation of the parabola shown below.



Since there is a repeated root, the equation is of the form:

$$y = k(x+5)(x+5)$$
$$= k(x+5)^2.$$

To find k, we use $(1, 6)$:

$$y = k(x+5)^2$$
$$6 = k(1+5)^2$$
$$k = \tfrac{6}{6^2} = \tfrac{1}{6}.$$

Hence $y = \frac{1}{6}(x+5)^2$.

When we know the turning point

Recall from Completing the Square that a parabola with turning point $(-p, q)$ has an equation of the form

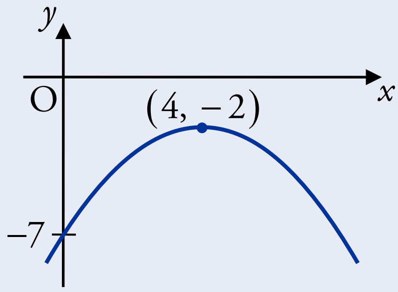
$$y = a(x+p)^2 + q$$

where a is some constant.

If we know another point on the parabola, then we can find the value of a.

EXAMPLE

3. Find the equation of the parabola shown below.



Since the turning point is $(4, -2)$, the equation is of the form:

$$y = a(x-4)^2 - 2.$$

To find a, we use $(0, -7)$:

$$y = a(x-4)^2 - 2$$
$$-7 = a(0-4)^2 - 2$$
$$16a = -5$$
$$a = -\tfrac{5}{16}.$$

Hence $y = -\frac{5}{16}(x-4)^2 - 2$.

6 Solving Quadratic Inequalities RC

The most efficient way of solving a quadratic inequality is by making a rough sketch of the parabola. To do this we need to know:

- the shape – concave up or concave down,
- the x-axis intercepts.

We can then solve the quadratic inequality by inspection of the sketch.

EXAMPLES

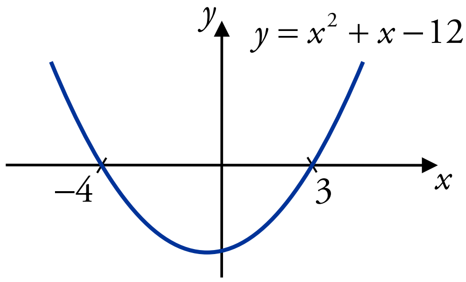
1. Solve $x^2 + x - 12 < 0$.

The parabola with equation $y = x^2 + x - 12$ is concave up.

The x-axis intercepts are given by:

$$x^2 + x - 12 = 0$$
$$(x+4)(x-3) = 0$$
$$x + 4 = 0 \quad \text{or} \quad x - 3 = 0$$
$$x = -4 \qquad\qquad x = 3.$$

Make a sketch:



So $x^2 + x - 12 < 0$ for $-4 < x < 3$.

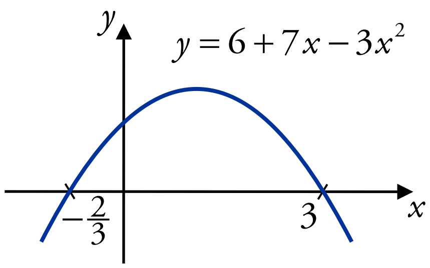
2. Find the values of x for which $6 + 7x - 3x^2 \geq 0$.

The parabola with equation $y = 6 + 7x - 3x^2$ is concave down.

The x-axis intercepts are given by:

$$6 + 7x - 3x^2 = 0$$
$$-\left(3x^2 - 7x - 6\right) = 0$$
$$(3x+2)(x-3) = 0$$
$$3x + 2 = 0 \quad \text{or} \quad x - 3 = 0$$
$$x = -\tfrac{2}{3} \qquad\qquad x = 3.$$

Make a sketch:



So $6 + 7x - 3x^2 \geq 0$ for $-\tfrac{2}{3} \leq x \leq 3$.

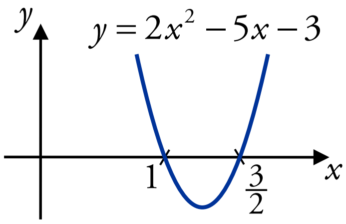
3. Solve $2x^2 - 5x - 3 > 0$.

The parabola with equation $y = 2x^2 - 5x - 3$ is concave up.

The x-axis intercepts are given by:

$$\begin{aligned} 2x^2 - 5x - 3 &= 0 \\ (x-1)(2x-3) &= 0 \\ x - 1 = 0 \quad &\text{or} \quad 2x - 3 = 0 \\ x = 1 \quad & \quad\quad\; x = \tfrac{3}{2}. \end{aligned}$$

Make a sketch:



So $2x^2 - 5x - 3 > 0$ for $x < 1$ and $x > \frac{3}{2}$.

4. Find the range of values of x for which the curve $y = \frac{1}{3}x^3 + 2x^2 - 5x + 3$ is strictly increasing.

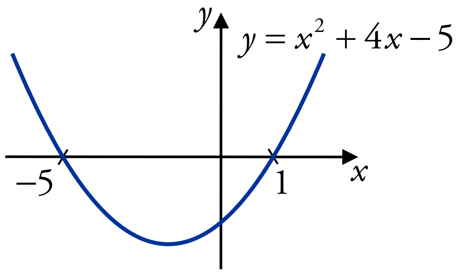
We have $\frac{dy}{dx} = x^2 + 4x - 5$.

Remember
Strictly increasing means $\frac{dy}{dx} > 0$.

The curve is strictly increasing where $x^2 + 4x - 5 > 0$.

$$\begin{aligned} x^2 + 4x - 5 &= 0 \\ (x-1)(x+5) &= 0 \\ x - 1 = 0 \quad &\text{or} \quad x + 5 = 0 \\ x = 1 \quad & \quad\quad\; x = -5. \end{aligned}$$

Make a sketch:



So the curve is strictly increasing for $x < -5$ and $x > 1$.

5. Find the values of q for which $x^2 + (q-4)x + \frac{1}{2}q = 0$ has no real roots.

For no real roots, $b^2 - 4ac < 0$:

$a = 1$

$b = q - 4$

$c = \frac{1}{2}q$

$$\begin{aligned} b^2 - 4ac &= (q-4)^2 - 4(1)\left(\tfrac{1}{2}q\right) \\ &= (q-4)(q-4) - 2q \\ &= q^2 - 8q + 16 - 2q \\ &= q^2 - 10q + 16. \end{aligned}$$

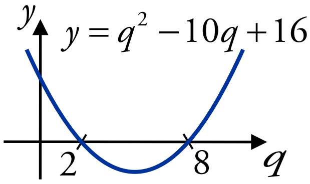
We now need to solve the inequality $q^2 - 10q + 16 < 0$.

The parabola with equation $y = q^2 - 10q + 16$ is concave up.

The x-axis intercepts are given by:

$$q^2 - 10q + 16 = 0$$
$$(q-2)(q-8) = 0$$
$$q - 2 = 0 \quad \text{or} \quad q - 8 = 0$$
$$q = 2 \qquad\qquad q = 8.$$

Make a sketch:

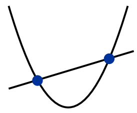


Therefore $b^2 - 4ac < 0$ for $2 < q < 8$, and so $x^2 + (q-4)x + \frac{1}{2}q = 0$ has no real roots when $2 < q < 8$.

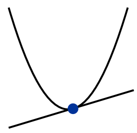
7 Intersections of Lines and Parabolas RC

To determine how many times a line intersects a parabola, we substitute the equation of the line into the equation of the parabola. We can then use the discriminant, or factorisation, to find the number of intersections.

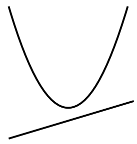
- If $b^2 - 4ac > 0$, the line and curve intersect twice.



- If $b^2 - 4ac = 0$, the line and curve intersect once (i.e. the line is a tangent to the curve).



- If $b^2 - 4ac < 0$, the line and the parabola do not intersect.



EXAMPLES

1. Show that the line $y = 5x - 2$ is a tangent to the parabola $y = 2x^2 + x$ and find the point of contact.

Substitute $y = 5x - 2$ into:

$$y = 2x^2 + x$$
$$5x - 2 = 2x^2 + x$$
$$2x^2 - 4x + 2 = 0$$
$$x^2 - 2x + 1 = 0$$
$$(x-1)(x-1) = 0.$$

Since there is a repeated root, the line is a tangent at $x = 1$.

To find the y-coordinate, substitute $x = 1$ into the equation of the line:

$$y = 5 \times 1 - 2 = 3.$$

So the point of contact is $(1, 3)$.

2. Find the equation of the tangent to $y = x^2 + 1$ that has gradient 3.

The equation of the tangent is of the form $y = mx + c$, with $m = 3$, i.e. $y = 3x + c$.

Substitute this into $y = x^2 + 1$

$$3x + c = x^2 + 1$$
$$x^2 - 3x + 1 - c = 0.$$

Since the line is a tangent:

$$b^2 - 4ac = 0$$
$$(-3)^2 - 4 \times (1 - c) = 0$$
$$9 - 4 + 4c = 0$$
$$4c = -5$$
$$c = -\tfrac{5}{4}.$$

Therefore the equation of the tangent is:

$$y = 3x - \tfrac{5}{4}$$
$$3x - y - \tfrac{5}{4} = 0.$$

Note
You could also do this question using methods from Differentiation.