Higher

Mathematics

Trigonometry

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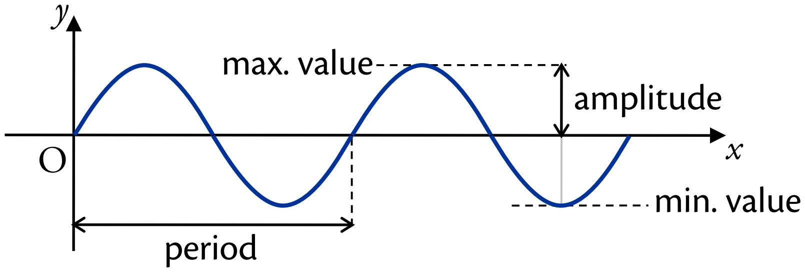
CfE Edition

7 Trigonometric Functions EF

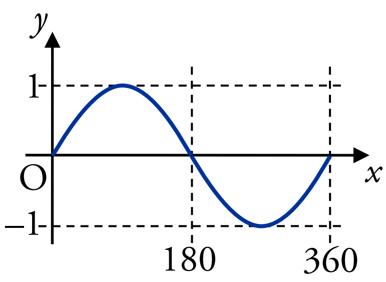
A function which has a repeating pattern in its graph is called **periodic**. The length of the smallest repeating pattern in the x-direction is called the **period**.

If the repeating pattern has a minimum and maximum value, then half of the difference between these values is called the **amplitude**.



The three basic trigonometric functions (sine, cosine, and tangent) are periodic, and have graphs as shown below.

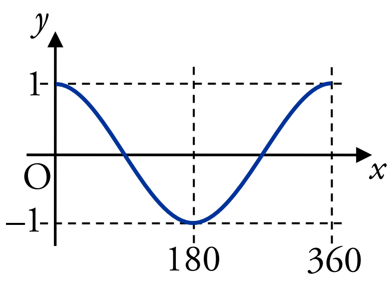
$y = \sin x°$



Period $= 360°$

Amplitude $= 1$

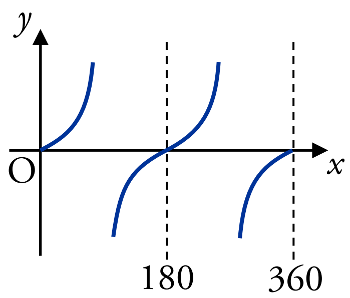
$y = \cos x°$



Period $= 360°$

Amplitude $= 1$

$y = \tan x°$



Period $= 180°$

Amplitude is undefined

Trigonometry

1 Radians EF

Degrees are not the only units used to measure angles. The radian (RAD on the calculator) is a measurement also used.

Degrees and radians bear the relationship:

$$\pi \text{ radians} = 180°.$$

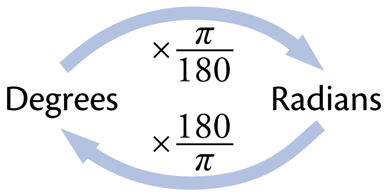
The other equivalences that you should become familiar with are:

$$30° = \tfrac{\pi}{6} \text{ radians} \qquad 45° = \tfrac{\pi}{4} \text{ radians} \qquad 60° = \tfrac{\pi}{3} \text{ radians}$$

$$90° = \tfrac{\pi}{2} \text{ radians} \qquad 135° = \tfrac{3\pi}{4} \text{ radians} \qquad 360° = 2\pi \text{ radians.}$$

Converting between degrees and radians is straightforward.

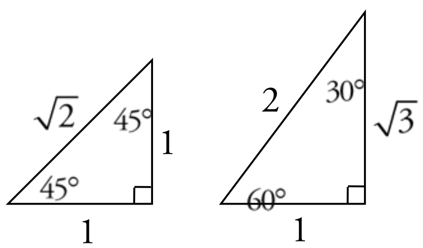
- To convert from degrees to radians, multiply by π and divide by 180.
- To convert from radians to degrees, multiply by 180 and divide by π.



For example, $50° = 50 \times \frac{\pi}{180} = \frac{5}{18}\pi$ radians.

2 Exact Values EF

The following exact values must be known. You can do this by either memorising the two triangles involved, or memorising the table.



DEG	RAD	$\sin x$	$\cos x$	$\tan x$
0	0	0	1	0
30	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90	$\frac{\pi}{2}$	1	0	–

Tip
You'll probably find it easier to remember the triangles.

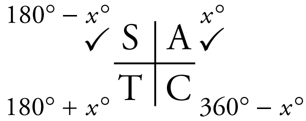
3 Solving Trigonometric Equations RC

You should already be familiar with solving some trigonometric equations.

EXAMPLES

1. Solve $\sin x° = \frac{1}{2}$ for $0 < x < 360$.

$\sin x° = \frac{1}{2}$



Since $\sin x°$ is positive

First quadrant solution:

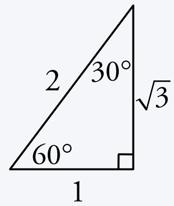
$$x = \sin^{-1}\left(\tfrac{1}{2}\right) = 30.$$

$x = 30$ or $180 - 30$

$x = 30$ or 150.

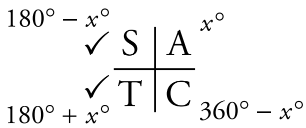
Remember

The exact value triangle:



2. Solve $\cos x° = -\frac{1}{\sqrt{5}}$ for $0 < x < 360$.

$\cos x° = -\frac{1}{\sqrt{5}}$



Since $\cos x°$ is negative

$$x = \cos^{-1}\left(\tfrac{1}{\sqrt{5}}\right) = 63{\cdot}435 \text{ (to 3 d.p.).}$$

$x = 180 - 63{\cdot}435$ or $180 + 63{\cdot}435$

$x = 116{\cdot}565$ or $243{\cdot}435$.

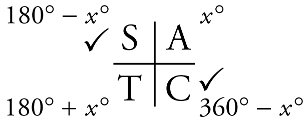
3. Solve $\sin x° = 3$ for $0 < x < 360$.

There are no solutions since $-1 \le \sin x° \le 1$.

Note that $-1 \le \cos x° \le 1$, so $\cos x° = 3$ also has no solutions.

4. Solve $\tan x° = -5$ for $0 < x < 360$.

$\tan x° = -5$



Since $\tan x°$ is negative

$$x = \tan^{-1}(5)$$
$$= 78{\cdot}690 \quad \text{(to 3 d.p.)}.$$

$x = 180 - 78{\cdot}690$ or $360 - 78{\cdot}690$

$x = 101{\cdot}310$ or $281{\cdot}310$.

Note

All trigonometric equations we will meet can be reduced to problems like those above. The only differences are:

- the solutions could be required in radians – in this case, the question will not have a degree symbol, e.g. "Solve $3\tan x = 1$" rather than "$3\tan x° = 1$";
- exact value solutions could be required in the non-calculator paper – you will be expected to know the exact values for 0, 30, 45, 60 and 90 degrees.

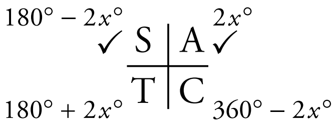
Questions can be worked through in degrees or radians, but make sure the final answer is given in the units asked for in the question.

EXAMPLES

5. Solve $2\sin 2x° - 1 = 0$ where $0 \le x \le 360$.

$2\sin 2x° = 1$

$\sin 2x° = \frac{1}{2}$



$0 \le x \le 360$

$0 \le 2x \le 720$

$$2x = \sin^{-1}\left(\tfrac{1}{2}\right)$$
$$= 30.$$

$2x = 30$ or $180 - 30$

or $360 + 30$ or $360 + 180 - 30$

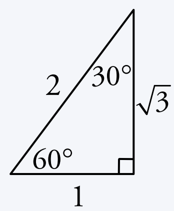
or ~~$360 + 360 + 30$~~

$2x = 30$ or 150 or 390 or 510

$x = 15$ or 75 or 195 or 255.

Remember

The exact value triangle:



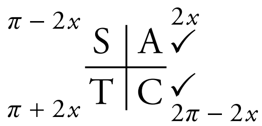
Note

There are more solutions every 360°, since $\sin(30°) = \sin(30°+360°) = \ldots$

So keep adding 360 until $2x > 720$.

6. Solve $\sqrt{2}\cos 2x = 1$ where $0 \leq x \leq \pi$.

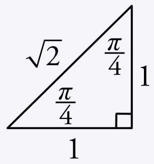
$\cos 2x = \frac{1}{\sqrt{2}}$



$0 \leq x \leq \pi$

$0 \leq 2x \leq 2\pi$

Remember
The exact value triangle:



$$2x = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}.$$

$2x = \frac{\pi}{4}$ or $2\pi - \frac{\pi}{4}$

or ~~$2\pi + \frac{\pi}{4}$~~

$2x = \frac{\pi}{4}$ or $\frac{7\pi}{4}$

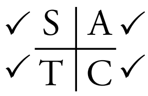
$x = \frac{\pi}{8}$ or $\frac{7\pi}{8}$.

7. Solve $4\cos^2 x = 3$ where $0 < x < 2\pi$.

$(\cos x)^2 = \frac{3}{4}$

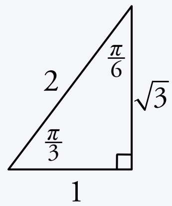
$\cos x = \pm\sqrt{\frac{3}{4}}$

$\cos x = \pm\frac{\sqrt{3}}{2}$



Since $\cos x$ can be positive or negative

Remember
The exact value triangle:



$$x = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}.$$

$x = \frac{\pi}{6}$ or $\pi - \frac{\pi}{6}$ or $\pi + \frac{\pi}{6}$ or $2\pi - \frac{\pi}{6}$

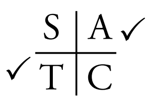
or ~~$2\pi + \frac{\pi}{6}$~~

$x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$ or $\frac{7\pi}{6}$ or $\frac{11\pi}{6}$.

8. Solve $3\tan(3x° - 20°) = 5$ where $0 \leq x \leq 360$.

$3\tan(3x° - 20°) = 5$

$\tan(3x° - 20°) = \frac{5}{3}$



$0 \leq x \leq 360$

$0 \leq 3x \leq 1080$

$-20 \leq 3x - 20 \leq 1060$

$$3x - 20 = \tan^{-1}\left(\frac{5}{3}\right) = 59{\cdot}036 \quad \text{(to 3 d.p.)}$$

$3x - 20 = 59{\cdot}036$ or $180 + 59{\cdot}036$

or $360 + 59{\cdot}036$ or $360 + 180 + 59{\cdot}036$

or $360 + 360 + 59{\cdot}036$ or $360 + 360 + 180 + 59{\cdot}036$

or ~~$360 + 360 + 360 + 59{\cdot}036$.~~

$$3x - 20 = 59{\cdot}036 \text{ or } 239{\cdot}036 \text{ or } 419{\cdot}036$$
$$\text{or } 599{\cdot}036 \text{ or } 779{\cdot}036 \text{ or } 959{\cdot}036$$
$$3x = 79{\cdot}036 \text{ or } 259{\cdot}036 \text{ or } 439{\cdot}036$$
$$\text{or } 619{\cdot}036 \text{ or } 799{\cdot}036 \text{ or } 979{\cdot}036$$
$$x = 26{\cdot}35 \text{ or } 86{\cdot}35 \text{ or } 146{\cdot}35 \text{ or } 206{\cdot}35 \text{ or } 266{\cdot}35 \text{ or } 326{\cdot}35.$$

9. Solve $\cos\left(2x + \frac{\pi}{3}\right) = 0{\cdot}812$ for $0 < x < 2\pi$.

$$\cos\left(2x + \tfrac{\pi}{3}\right) = 0{\cdot}812$$

S	A ✓
T	C ✓

$$0 < x < 2\pi$$
$$0 < 2x < 4\pi$$
$$\tfrac{\pi}{3} < 2x + \tfrac{\pi}{3} < 4\pi + \tfrac{\pi}{3}$$
$$1{\cdot}047 < 2x + \tfrac{\pi}{3} < 13{\cdot}614 \quad \text{(to 3 d.p.)}$$

$$2x + \tfrac{\pi}{3} = \cos^{-1}(0{\cdot}812)$$
$$= 0{\cdot}623 \quad \text{(to 3 d.p.)}.$$

Remember
Make sure your calculator uses radians

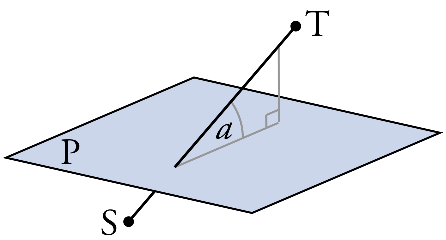
$$2x + \tfrac{\pi}{3} = \cancel{0{\cdot}623} \text{ or } 2\pi - 0{\cdot}623$$
$$\text{or } 2\pi + 0{\cdot}623 \text{ or } 2\pi + 2\pi - 0{\cdot}623$$
$$\text{or } 2\pi + 2\pi + 0{\cdot}623 \text{ or } \cancel{2\pi + 2\pi + 2\pi - 0{\cdot}623}$$
$$2x + \tfrac{\pi}{3} = 5{\cdot}660 \text{ or } 6{\cdot}906 \text{ or } 11{\cdot}943 \text{ or } 13{\cdot}189$$
$$2x = 4{\cdot}613 \text{ or } 5{\cdot}859 \text{ or } 10{\cdot}896 \text{ or } 12{\cdot}142$$
$$x = 2{\cdot}307 \text{ or } 2{\cdot}930 \text{ or } 5{\cdot}448 \text{ or } 6{\cdot}071.$$

4 Trigonometry in Three Dimensions EF

It is possible to solve trigonometric problems in three dimensions using techniques we already know from two dimensions. The use of sketches is often helpful.

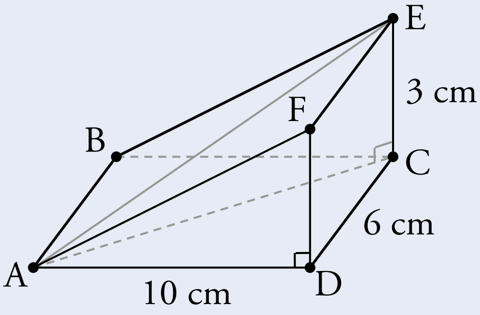
The angle between a line and a plane

The angle a between the plane P and the line ST is calculated by adding a line perpendicular to the plane and then using basic trigonometry.



EXAMPLE

1. The triangular prism ABCDEF is shown below.

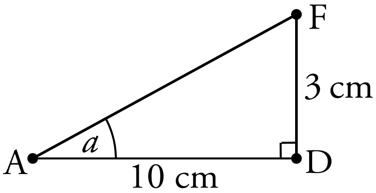


Calculate the acute angle between:

(a) The line AF and the plane ABCD.

(b) AE and ABCD.

(a) Start with a sketch:



$$\tan a = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{3}{10}$$

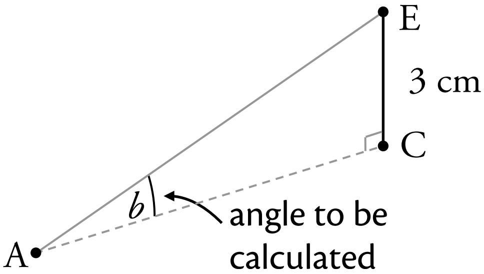
$$a = \tan^{-1}\left(\frac{3}{10}\right)$$

$$= 16{\cdot}699° \quad \text{(or } 0{\cdot}291 \text{ radians)} \quad \text{(to 3 d.p.).}$$

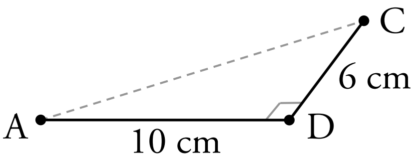
Note

Since the angle is in a right-angled triangle, it must be acute so there is no need for a CAST diagram.

(b) Again, make a sketch:



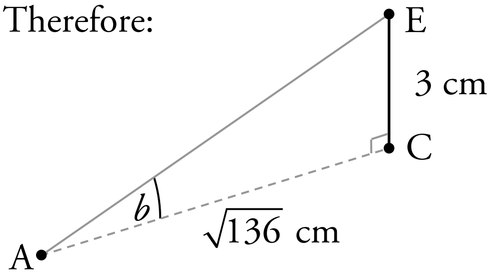
We need to calculate the length of AC first using Pythagoras's Theorem:



$$AC = \sqrt{10^2 + 6^2}$$

$$= \sqrt{136}$$

Therefore:



$$\tan b = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{3}{\sqrt{136}}$$

$$b = \tan^{-1}\left(\frac{3}{\sqrt{136}}\right)$$

$$= 14{\cdot}426° \quad \text{(or } 0{\cdot}252 \text{ radians)} \quad \text{(to 3 d.p.).}$$