Further Calculus

Remember:

- $f(x) = ax^n \implies f'(x) = anx^{n-1}$ where n is a rational number.
- y is the same as $f(x)$ and $\frac{dy}{dx}$ is the same as $f'(x)$.
- The Derivative of a constant is zero.
- Differentiating gives the **GRADIENT** of a curve at the point $x = a$.
- Prior to Differentiating a function the following must be true:
    - All brackets should be multiplied out.
    - Roots need to be changed to powers: $\sqrt[m]{x^n} = x^{n/m}$
    - x cannot appear in the denominator of a fraction: $\frac{a}{bx^n} = \frac{a}{b}x^{-n}$
- After Differentiating it is good practice to return the expression to the form the question gave it in.

Differentiating Trig Functions:

- $y = \sin\theta \implies \frac{dy}{d\theta} = \cos\theta$
- $y = \cos\theta \implies \frac{dy}{d\theta} = -\sin\theta$

← **Given on your Formulae Sheet**

- For these results to be valid the angle **must** be measured in **RADIANS**.
  This is why we usually use θ rather than x

Examples:

1. Differentiate the following with respect to θ:

a) $y = 5\sin\theta$ $\qquad \frac{dy}{d\theta} = 5\cos\theta$

b) $y = 3\sin\theta + 2\cos\theta$ $\qquad \frac{dy}{d\theta} = 3\cos\theta - 2\sin\theta$

2. For the function in part **(b)** above find $\frac{dy}{d\theta}$ when $\theta = \frac{\pi}{6}$

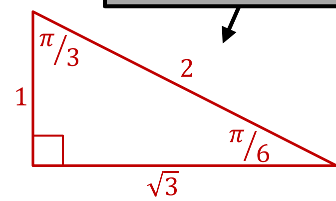
$$\frac{dy}{d\theta} = 3\cos\theta - 2\sin\theta \implies 3\cos\frac{\pi}{6} - 2\sin\frac{\pi}{6}$$

$$\implies 3 \times \frac{\sqrt{3}}{2} - 2 \times \frac{1}{2}$$

$$\implies \frac{3\sqrt{3}}{2} - 1$$

$$\implies \frac{3\sqrt{3}-2}{2}$$

Must know the 2 excat value triangles and 3 Trig graphs!



3. Find $f'(x)$ when $f(x) = \frac{1 - x\cos x}{3x}$

$$f(x) = \frac{1 - x\cos x}{3x}$$

$$f(x) = \frac{1}{3x} - \frac{x\cos x}{3x}$$

$$f(x) = 3x^{-1} - \frac{1}{3}\cos x$$

$$f'(x) = -3x^{-2} - \left(-\frac{1}{3}\sin x\right)$$

$$f'(x) = -\frac{3}{x^2} + \frac{1}{3}\sin x$$

Split into separate fractions before Differentiating.

4. For the function in part **(a)** if question **1** above find the equation of the tangent when $\theta = \frac{\pi}{4}$

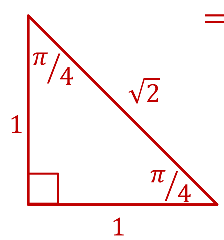
$$y = 5\sin\theta \implies \frac{dy}{d\theta} = 5\cos\theta$$

GRADIENT

$$\text{when } \theta = \frac{\pi}{4} \implies \frac{dy}{d\theta} = 5\cos\frac{\pi}{4}$$

$$\implies \frac{dy}{d\theta} = 5 \times \frac{1}{\sqrt{2}}$$

$$\implies \frac{dy}{d\theta} = \frac{5}{\sqrt{2}}$$



POINT

$$\text{when } \theta = \frac{\pi}{4} \implies y = 5\sin\frac{\pi}{4}$$

$$\implies y = 5 \times \frac{1}{\sqrt{2}}$$

$$\implies y = \frac{5}{\sqrt{2}} \qquad \left(\frac{\pi}{4}, \frac{5}{\sqrt{2}}\right)$$

EQUATION

$$y - b = m(x - a)$$

$$y - \frac{5}{\sqrt{2}} = \frac{5}{\sqrt{2}}\left(x - \frac{\pi}{4}\right)$$

$$\sqrt{2}y - 5 = 5x - \frac{5\pi}{4}$$

$$\sqrt{2}y - 5x = 5 - \frac{5\pi}{4}$$

$$\sqrt{2}y - 5x = \frac{20 - 5\pi}{4}$$

Chain Rule:

- The CHAIN RULE is a method of Differentiating composite functions.
- We usually concentrate on functions in the form: $y = (f(x))^n \implies \frac{dy}{dx} = n\left(f(x)\right)^{n-1} \times f'(x)$
- A simple way of thinking about the Chain Rule is:

"*Differentiate outside the bracket multiplyed by the derivative of inside the bracket*"

Examples:

OUTSIDE INSIDE

5. Differentiate the following with respect to *x* :

a) $y = (7x^2 - 3x + 5)^5$

$\frac{dy}{dx} = 5 \times (7x^2 - 3x + 5)^4 \times (14x - 3)$

$\frac{dy}{dx} = 5(14x - 3)(7x^2 - 3x + 5)^4$

b) $y = \frac{1}{(4x+5)^2},\ x > 0$

$y = (4x + 5)^{-2} \implies \frac{dy}{dx} = -2 \times (4x + 5)^{-3} \times 4$

$\implies \frac{dy}{dx} = \frac{-8}{(4x+5)^3}$

c) $y = \sqrt[4]{(2x - 8)^3},\ x \neq -4$

$y = (2x - 8)^{3/4} \implies \frac{dy}{dx} = \frac{3}{4} \times (2x - 8)^{-1/4} \times 2$

$\implies \frac{dy}{dx} = \frac{3}{2} \times \frac{1}{(2x-8)^{1/4}}$

$\implies \frac{dy}{dx} = \frac{3}{2\sqrt[4]{(2x-8)}}$

Chain Rule for Trig:

- Remember that $y = \sin^n\theta$ actually means $y = (\sin\theta)^n$, similarly for cos so we can use the Chain Rule!
- The Chain Rule can also be used for Differentiating Trig Functions as follows:
  - $y = \sin(ax + b) \implies \frac{dy}{dx} = a\cos(ax + b)$
  - $y = \cos(ax + b) \implies \frac{dy}{dx} = -a\sin(ax + b)$

Examples:

5. Differentiate the following with respect to *x* :

a) $y = 2\cos^4 x$

$y = 2(\cos x)^6 \quad \frac{dy}{dx} = 6 \times 2(\cos x)^5 \times -\sin x$

$\frac{dy}{dx} = -12\cos^5 x \sin x$

b) $y = \sin\left(7x + \frac{\pi}{3}\right)$

$\frac{dy}{dx} = \cos\left(7x + \frac{\pi}{3}\right) \times 7$

$\frac{dy}{dx} = 7\cos\left(7x + \frac{\pi}{3}\right)$

Remember:

Must include!

- $\int ax^n \; dx \; = \frac{ax^{n+1}}{n+1} + c$ where c is the constant of integration.
- The Integral of a constant is x.
- Integrating gives the **Area Under the Curve** . These are called DEFINITE INTEGRALS.
- Prior to Integrating a function the following must be true:
    - All brackets should be multiplied out.
    - Roots need to be changed to powers: $\sqrt[m]{x^n} = x^{n/m}$
    - x cannot appear in the denominator of a fraction: $\frac{a}{bx^n} = \frac{a}{b}x^{-n}$
- After Integrating it is good practice to return the expression to the form the question gave it in.

A Special Integral:

- A function in the form $y = (ax + b)^n$ can be integrated as follows: $\int (ax+b)^n \, dx \; = \frac{(ax+b)^{n+1}}{a(n+1)} + c$

Examples:

6. Integrate the following:

a) $\int (4x-7)^4 \, dx$

$$= \frac{(4x-7)^5}{4 \times 5} + c \quad = \frac{(4x-7)^5}{20} + c$$

b) $\int \frac{6}{\sqrt[3]{3x+2}} dx \;,\; x > 0$

$$= \int 6(3x+2)^{-1/3} \, dx = \frac{6(3x+2)^{2/3}}{2/3 \times 3}$$

$$= \frac{6\sqrt[3]{(3x+2)^2}}{2}$$

$$= 3\sqrt[3]{(3x+2)^2}$$

c) $\int_0^4 \sqrt{3x+4} \, dx \;,\; x \geq 0$

$$= \int_0^4 (3x+4)^{1/2} \, dx \quad = \left[\frac{(3x+4)^{3/2}}{3/2 \times 3}\right]_0^4$$

$$= \left[\frac{2\sqrt{(3x+4)^3}}{9}\right]_0^4$$

$$= \frac{2\sqrt{(3(4)+4)^3}}{9} - \frac{2\sqrt{(3(0)+4)^3}}{9}$$

$$= \frac{2\sqrt{16^3}}{9} - \frac{2\sqrt{4^3}}{9}$$

$$= \frac{2 \times 4^3}{9} - \frac{2 \times 2^3}{9}$$

$$= \frac{128}{9} - \frac{16}{9}$$

$$= \frac{112}{9}$$

Integrating Trig Functions:

- Trig Functions can be Integrated as follows:
  - $\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b)+c$
  - $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b)+c$

Given on your Formulae Sheet

Examples:

7. Integrate the following:

a) $\int \sin(3x-1)\,dx$

$= -\frac{1}{3}\cos(3x-1)+c$

b) $\int 3\cos\left(\frac{3}{4}x+2\right)dx$

$= 3 \times \frac{4}{3}\sin\left(\frac{3}{4}x+2\right)+c$

$= 4\sin\left(\frac{3}{4}x+2\right)+c$

1 over a fraction, FLIP the bottom fraction

c) $\int 5\cos 2x + \sin\left(x-\sqrt{3}\right)dx$

$= \frac{1}{2} \times 5\sin 2x - \cos\left(x-\sqrt{3}\right)+c$

$= \frac{5}{2}\sin 2x - \cos\left(x-\sqrt{3}\right)+c$

d) $\int 5\cos^2 x\,dx$

We cannot integrate squared Trig functions directly. First change it using an expansion from eralier in the course.

$\cos 2x = 2\cos^2 x - 1$

$\cos 2x + 1 = 2\cos^2 x$

$\cos^2 x = ½\cos 2x + ½$

$= \int 5\left(\frac{1}{2}\cos 2x + \frac{1}{2}\right)dx$

$= \int \frac{5}{2}\cos 2x + \frac{5}{2}\,dx$

$= \frac{5}{2} \times \frac{1}{2}\sin 2x + \frac{5}{2}x + c$

$= \frac{5}{4}\sin 2x + \frac{5}{2}x + c$

e) $\int \sin x \cos x\,dx$

We cannot integrate products of Trig functions directly. First change it using an expansion from earlier.

$\sin 2x = 2\sin x\cos x$

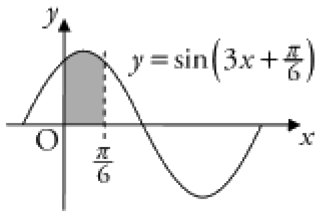
$½\sin 2x = \sin x\cos x$

$= \int \frac{1}{2}\sin 2x\,dx$

$= \frac{1}{2} \times -\frac{1}{2}\cos 2x + c$

$= -\frac{1}{4}\cos 2x + c$

8. Find this shaded area:



$$\int_0^{\frac{\pi}{6}} \sin\left(3x + \frac{\pi}{6}\right) dx$$

$$= \left[-\frac{1}{3}\cos\left(3x + \frac{\pi}{6}\right)\right]_0^{\frac{\pi}{6}}$$

$$= -\frac{1}{3}\cos\left(3\left(\frac{\pi}{6}\right) + \frac{\pi}{6}\right) - \left(-\frac{1}{3}\cos\left(3(0) + \frac{\pi}{6}\right)\right)$$

$$= -\frac{1}{3}\cos\left(\frac{3\pi}{6} + \frac{\pi}{6}\right) - \left(-\frac{1}{3}\cos\left(\frac{\pi}{6}\right)\right)$$

$$= -\frac{1}{3}\cos\left(\frac{4\pi}{6}\right) + \frac{1}{3}\cos\left(\frac{\pi}{6}\right)$$

$$= -\frac{1}{3}\times\left(-\frac{1}{2}\right) + \frac{1}{3}\times\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{1}{6} + \frac{\sqrt{3}}{6}$$

$= \frac{1+\sqrt{3}}{6}$ so the Area is $\frac{1+\sqrt{3}}{6}$ units^2

9. Find the Particular Solution of the Differential Equation $\frac{dy}{dx} = 3\sin 3x$ given that $y = 2$ when $x = \frac{\pi}{3}$

Integrate the function and then use the values to find C as seen earlier in the course

$y = \int 3\sin 3x \quad \Rightarrow y = 3 \times -\frac{1}{3}\cos 3x + c$

$\Rightarrow y = -\cos 3x + c$ ← **General Solution**

When $x = \frac{\pi}{3}, y = 2 \quad \Rightarrow 2 = -\cos 3\left(\frac{\pi}{3}\right) + c$

$\Rightarrow 2 = -\cos(\pi) + c$

$\Rightarrow 2 = -(-1) + c$

$\Rightarrow 2 = 1 + c$

$\Rightarrow c = 1$

$\Rightarrow y = -\cos 3x + 1$ ← **Particular Solution**