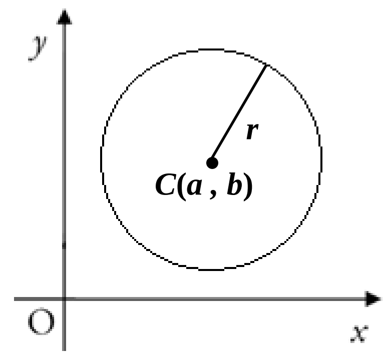
H The Circle

SPTA Mathematics – Higher Notes

Equation of a Circle:

- To find the equation of a circle you need 2 things:
    - Coordinates of the centre, (a , b)
    - The radius, r



- The equation of a Circle is given by: $(x-a)^2+(y-b)^2=r^2$
- The equation of a Circle with centre the origin is given by: $x^2+y^2=r^2$

You do not need to expand the brackets unless the question specifies to do so but you must always square the radius!!

- You will need to remember these two formulae covered in the STRAIGH LINE topic earlier:
    - Distance Formula: $\sqrt{(x_B-x_A)^2+(y_B-y_A)^2}$
    - Midpoint Formula: $\left(\frac{x_A+x_B}{2} , \frac{y_A+y_B}{2}\right)$

Must know!!

Examples:

1. Write down the equation of the circle with:

a) Centre (0 , 0) and radius = 6:

$$x^2+y^2=6^2$$
$$\Rightarrow \quad x^2+y^2=36$$

b) Centre (5 , –2) and radius = 3.5:

$$(x-5)^2+(y-(-2))^2=3.5^2$$
$$\Rightarrow \quad (x-5)^2+(y+2)^2=12.25$$

c) Centre (3 , 0) and radius = $4\sqrt{3}$:

$$(x-3)^2+(y-0)^2=(4\sqrt{3})^2$$
$$\Rightarrow \quad (x-3)^2+y^2=(\sqrt{48})^2$$
$$\Rightarrow \quad (x-3)^2+y^2=48$$

d) Centre (3 , 6) passing through (–1 , 4) :

$\text{Radius} = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$

$\text{Radius} = \sqrt{(-1-3)^2 + (4-6)^2}$

$\text{Radius} = \sqrt{(-4)^2 + (-2)^2}$

$\text{Radius} = \sqrt{16+4}$

$\text{Radius} = \sqrt{20}$

No need to square root it as going to square it for the equation!!

so $(x-3)^2 + (y-6)^2 = (\sqrt{20})^2$

$(x-3)^2 + (y-6)^2 = 20$

2. State the Centre and Radius of these circles:

a) $(x-8)^2 + (y+3)^2 = 36$ Centre (8 , –3) & Radius = $\sqrt{36} = 6$

b) $x^2 + y^2 = 144$ Centre (0 , 0) & Radius = $\sqrt{144} = 12$

c) $(x+7)^2 + (y-4)^2 = 18$ Centre (–7 , 4) & Radius = $\sqrt{18} = 3\sqrt{2}$

Normally leave as a SURD

3. Find the equation of the circle when the points *A*(-5 , 3) and *B*(3 , 1) are the endpoints of the diameter.

Midpoint $AB = \left(\frac{x_A+x_B}{2} , \frac{y_A+y_B}{2}\right)$

Midpoint $AB = \left(\frac{-5+3}{2} , \frac{3+1}{2}\right)$

Midpoint $AB = (-1 , 2)$

$\text{Radius} = \sqrt{(x_M - x_A)^2 + (y_M - y_A)^2}$

$\text{Radius} = \sqrt{(-1-(-5))^2 + (2-3)^2}$

$\text{Radius} = \sqrt{4^2 + (-1)^2}$

$\text{Radius} = \sqrt{16+1}$

$\text{Radius} = \sqrt{17}$

so $(x-(-1))^2 + (y-2)^2 = (\sqrt{17})^2$

$(x+1)^2 + (y-2)^2 = 17$

4. **a)** Prove that the triangle ABC is tight angled when $A(-3\ ,-2)$, $B(3\ ,-1)$ and $C(2\ ,5)$,

$AB = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$

$AB = \sqrt{(3-(-3))^2 + (-1-(-2))^2}$

$AB = \sqrt{6^2 + 1^2}$

$AB = \sqrt{36+1}$

$AB = \sqrt{37}$

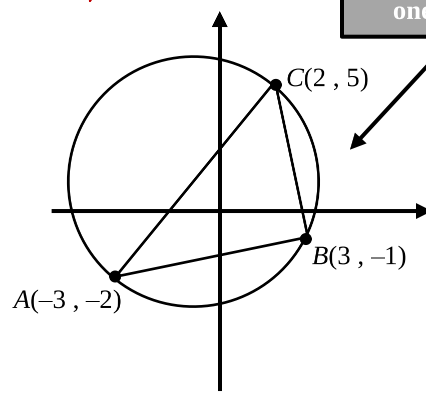
$AC = \sqrt{(2-(-3))^2 + (5-(-2))^2}$

$AC = \sqrt{5^2 + 7^2}$

$AC = \sqrt{25+49}$

$AC = \sqrt{74}$

If no diagram is given it is always useful to sketch one anyway!!!



$BC = \sqrt{(2-3)^2 + (5-(-1))^2}$

$BC = \sqrt{(-1)^2 + 6^2}$

$BC = \sqrt{1+36}$

$BC = \sqrt{37}$

Since $AC^2 = AB^2 + BC^2$ the triangle is right angled.

Angle in a semicircle from Nat 5

b) Hence find the equation of the circle passing through these 3 points.

From part (a), we can see that AC is the diameter of the circle.

Midpoint $AC = \left(\frac{x_A + x_C}{2}\ ,\ \frac{y_A + y_C}{2}\right)$

Midpoint $AC = \left(\frac{-3+2}{2}\ ,\ \frac{-2+5}{2}\right)$

Midpoint $AC = \left(-\frac{2}{2}\ ,\ \frac{3}{2}\right)$

Radius $= \sqrt{(x_M - x_A)^2 + (y_M - y_A)^2}$

Radius $= \sqrt{(-1-(-5))^2 + (2-3)^2}$

Radius $= \sqrt{4^2 + (-1)^2}$

Radius $= \sqrt{16+1}$

Radius $= \sqrt{17}$

so $(x-(-1))^2 + (y-2)^2 = (\sqrt{17})^2$

$(x+1)^2 + (y-2)^2 = 17$

General Equation of a Circle:

- The equation of a circle can also be given in the following form:

  $x^2 + y^2 + 2gx + 2fy + c = 0$ where the centre is $(-g, -f)$ and the radius is $\sqrt{g^2 + f^2 - c}$

- As soon as you see an equation of a circle in a question it is **ALWAYS** worthwhile stating the centre & radius, regardless of what format the equation is given in or what the question is asking!!

Examples:

5. Find the radius and centre of: $x^2 + y^2 + 4x - 8y + 7 = 0$

$2g = 4$ $\quad 2f = -8$ $\quad c = 7$

$g = 2$ $\quad f = -4$ $\quad$ so centre is $(-2, 4)$

Radius $= \sqrt{g^2 + f^2 - c}$

Radius $= \sqrt{2^2 + (-4)^2 - 7}$

You should use g and f NOT the centre

Radius $= \sqrt{4 + 16 - 7}$

Radius $= \sqrt{13}$ units

6. Find the radius and centre of: $2x^2 + 2y^2 - 6x + 10y - 2 = 0$

$x^2 + y^2 - 3x + 5y - 1 = 0$

The equation must be in the form: $x^2 + y^2 + 2gx + 2fy + c = 0$ so divide the original by 2

$2g = -3$ $\quad 2f = 5$ $\quad c = -1$

$g = \frac{-3}{2}$ $\quad f = \frac{5}{2}$

so centre is $\left(\frac{3}{2}, \frac{-5}{2}\right)$

Radius $= \sqrt{g^2 + f^2 - c}$

Radius $= \sqrt{\left(\frac{-3}{2}\right)^2 + \left(\frac{5}{2}\right)^2 - (-1)}$

Radius $= \sqrt{\frac{9}{4} + \frac{25}{4} + 1}$

Radius $= \sqrt{\frac{38}{4}} = \frac{\sqrt{38}}{2}$ units

7. Explain why $x^2 + y^2 - 4x + 2y + 42 = 0$ is not an equation of a circle.

$2g = -4 \qquad 2f = 2 \qquad c = 42$

$g = -2 \qquad f = 1$

so centre is $(2, -1)$

Did not require the centre for this question.

Radius $= \sqrt{g^2 + f^2 - c}$

Radius $= \sqrt{(-2)^2 + 1^2 - 42}$

Radius $= \sqrt{4 + 1 - 42}$

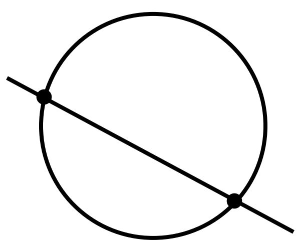
Radius $= \sqrt{-37}$

since root of a negative is not possible, the equation above does not represent a circle.

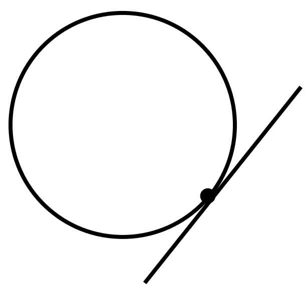
This is not totally accurate, but at higher it is fine!!

Intersection of Circles with Lines:

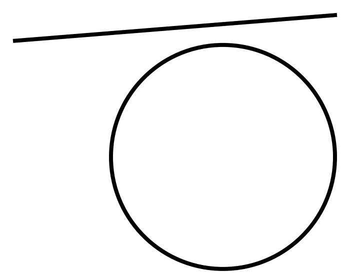
- Straight lines can intersect (meet) circles at either 2, 1 or zero points as follows:



2 Points of intersection
$b^2 - 4ac > 0$



1 Point of intersection
So Tangent to the circle
$b^2 - 4ac = 0$



No Points of intersection
$b^2 - 4ac < 0$

Discriminant from National 5

- To determine the nature of the points of intersection we use the SUBSTITUTION method of Simultaneous equations from Naionalt 5.

Examples:

8. Find where the line $y = 4x$ meets the circle with equation $x^2 + y^2 = 34$

Substitute $y = 4x$ into the circle $x^2 + y^2 = 24 \implies x^2 + (4x)^2 = 34$

$$\implies x^2 + 16x^2 = 34$$
$$\implies 17x^2 = 34$$
$$\implies x^2 = 2$$
$$\implies x = \pm\sqrt{2}$$

When $x = \pm\sqrt{2}$: $y = \pm 4\sqrt{2}$ So Points of Intersection are: $(\sqrt{2}, 4\sqrt{2})$ and $(-\sqrt{2}, -4\sqrt{2})$

9. Find the points where the line $2x + y - 6 = 0$ intersects the circle $x^2 + y^2 - 2x + 2y - 8 = 0$

$2x + y - 6 = 0 \implies y = -2x + 6$ Substitute $y = -2x + 6$ into the circle :

$$x^2 + y^2 - 2x + 2y + 8 = 0 \implies x^2 + (-2x + 6)^2 - 2x + 2(-2x + 6) - 8 = 0$$
$$\implies x^2 + (4x^2 - 24x + 36)^2 - 2x - 4x + 12 - 8 = 0$$
$$\implies 5x^2 - 30x + 40 = 0$$
$$\implies x^2 - 6x + 8 = 0$$
$$\implies (x - 2)(x - 4) = 0$$
$$\implies \mathrm{x} - 2 = 0 \quad x - 4 = 0$$
$$\implies x = 2,\ 4$$

When $x = 2$: $y = -2(2) + 6 \implies y = 2$ so Pt $(2, 2)$

When $x = 4$: $y = -2(4) + 6 \implies y = -2$ so Pt $(4, -2)$

10. Prove that the line $x + y = 4$ is a tangent to the circle $x^2 + y^2 + 6x + 2y - 22 = 0$ and find the point of contact.

$x + y = 4 \Longrightarrow \quad y = -x + 4$ Substitute $y = -x + 4$ into the circle :

$$x^2 + y^2 + 6x + 2y - 22 = 0 \Longrightarrow x^2 + (-x + 4)^2 + 6x + 2(-x + 4) - 22 = 0$$
$$\Longrightarrow x^2 + (x^2 - 8x + 16)^2 + 6x - 2x + 8 - 22 = 0$$
$$\Longrightarrow 2x^2 - 4x + 2 = 0$$
$$\Longrightarrow x^2 - 2x + 1 = 0$$
$$\Longrightarrow (x-1)(x-1) = 0$$
$$\Longrightarrow x - 1 = 0$$
$$\Longrightarrow x = 1$$

The Discriminant could be used at this point as shown below →

When $x = 1$: $y = -1 + 4 \Longrightarrow y = 3$ so Pt (1 , 3)

Since only one point of contact the line is a tangent to the circle at the point (1 , 3)

OR using the **DISCRIMINANT**:

$x^2 - 2x + 1 = 0$ so $a = 1,\ b = -2,\ c = 1$

$$b^2 - 4ac = (-2)^2 - 4(1)(1)$$
$$= 4 - 4$$
$$= 0$$

since $b^2 - 4ac = 0$ the line is a tangent to the circle.

$$x^2 - 2x + 1 = 0$$
$$\Longrightarrow (x-1)(x-1) = 0$$
$$\Longrightarrow x - 1 = 0$$
$$\Longrightarrow x = 1$$

When $x = 1$: $y = -1 + 4 \Longrightarrow y = 3$ so Pt (1 , 3)

In the example above it is probably more sensible to use the first method as you need to find the Point of Intersection anyway. If you are asked simply to prove that the Line is a Tangent then either method is suitable.

11. The line with equation $x - 3y = k$ is a tangent to the circle $x^2 + y^2 - 6x + 8y + 15 = 0$
Find the possible values of k

Change to $x =$ to avoid fractions

$x - 3y = k \implies x = 3y + k$ Substitute $x = 3y + k$ into the circle :

$$x^2 + y^2 - 6x + 8y + 15 = 0 \implies (3y + k)^2 + y^2 - 6(3y + k) + 8y + 15 = 0$$
$$\implies 9y^2 + 6ky + k^2 + y^2 - 18y - 6k + 8y + 15 = 0$$
$$\implies 10y^2 + 6ky - 10y + k^2 - 6k + 15 = 0$$
$$\implies 10y^2 + (6k - 10)y + (k^2 - 6k + 15) = 0$$

so $a = 10,\ b = 6k - 10,\ c = k^2 - 6k + 15$

Since the line is a tangent $b^2 - 4ac = 0$

$$\implies (6k - 10)^2 - 4(10)(k^2 - 6k + 15) = 0$$
$$\implies 36k^2 - 120k + 100 - 40k^2 + 240k - 600 = 0$$
$$\implies -4k^2 + 120k - 500 = 0$$
$$\implies k^2 - 30k + 125 = 0$$
$$\implies (k - 5)(k - 25) = 0$$
$$\implies k - 5 = 0 \quad k - 25 = 0$$
$$\implies k = 5,\ 25$$

Equation of a Tangent to a Circle:

- You can find the equation of a tangent to a circle if you know the point of contact as follows:
  - Find the centre of the circle
  - Find the gradient of the radius to the point of contact, m_{RAD},
  - Find the perpendicular gradient for the tangent using, $m_{TAN} \times m_{RAD} = -1$
  - Use $y - b = m(x - a)$ to find the tangent's equation.

Examples:

12. a) Prove that the point $A(2\,,-1)$ lies on the circle $x^2 + y^2 + 8x - 3y - 24 = 0$

b) Find the equation of the tangent to the circle at this point.

a) Sub the point into the circle equation: $2^2 + (-1)^2 + 8(2) - 3(-1) - 24$

$= 4 + 1 + 16 + 3 - 24$

$= 0$ hence point lies on the circle.

b) $2g = 8$ $\quad 2f = -3$ $\quad c = -24$

$g = 4$ $\quad f = -\frac{3}{2}$ $\quad$ so centre is $C(4\,, -\frac{3}{2})$

$$\text{Radius} = \sqrt{g^2 + f^2 - c}$$

$$\text{Radius} = \sqrt{4^2 + \left(\frac{-3}{2}\right)^2 + 24}$$

$$\text{Radius} = \sqrt{16 + \frac{9}{4} + 24}$$

$$\text{Radius} = \sqrt{42\frac{1}{4}}$$

Did not require the radius for this question.

$$m_{AC} = \frac{y_C - y_A}{x_C - x_A} = \frac{4-2}{-^3/_2 - (-1)} = \frac{2}{-^1/_2} = -4$$

Since perpendicular

$m_{Rad} \times m_{Tan} = -1$

so $m_{Tan} = \frac{1}{4}$

$y - b = m(x - a)$

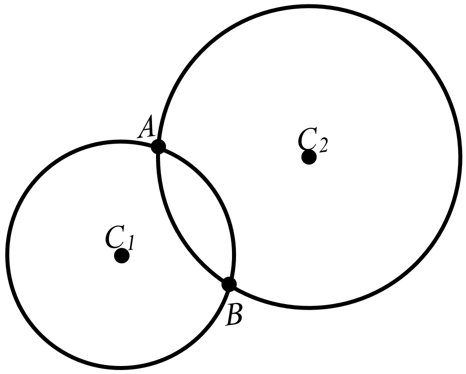
$y - 2 = \frac{1}{4}(x - (-1))$

$4y - 8 = x + 1$

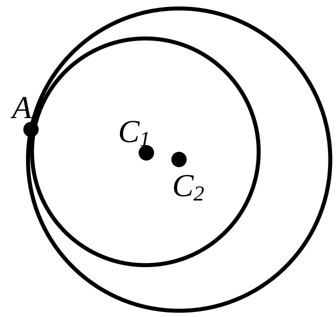
$4y = x + 9$

Intersection of Circles:

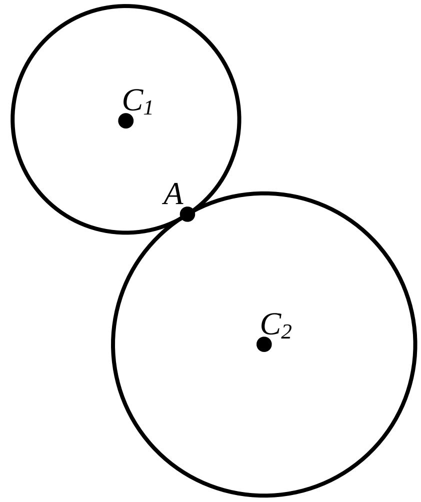
- 2 Circles can touch or intersect in 3 ways:



2 Points of intersection
Circles cross over.
Distance between centres is less than sum of the 2 Radii.



1 Point of intersection
Circles touch **INTERNALLY**
Distance between centres equals difference of the 2 Radii.



1 Point of intersection
Circles touch **EXTERNALLY**
Distance between centres equals sum of the 2 Radii.

- To prove they don’t touch then the distance between centres is greater than the sum of the Radii.

Examples:

13. Circle *P* has centre (–4 , –1) and radius 2 units.

Circle Q has equation $x^2 + y^2 - 2x + 6y + 1 = 0$.

Show that circles *P* and *Q* do not touch.

$2g = -2 \qquad 2f = 6 \qquad c = 1$

$g = -1 \qquad f = 3 \qquad$ so centre is $C(1, -3)$

$\text{Radius} = \sqrt{g^2 + f^2 - c}$

$\text{Radius} = \sqrt{(-1)^2 + 3^2 - 1}$

$\text{Radius} = \sqrt{9} = 3$

$\text{Centres} = \sqrt{(x_Q - x_P)^2 + (y_Q - y_P)^2}$

$\text{Centres} = \sqrt{(1 - (-4))^2 + (-3 - (-1))^2}$

$\text{Centres} = \sqrt{5^2 + (-2)^2}$

$\text{Centres} = \sqrt{25 + 4} = \sqrt{29}$

Since $R_1 + R_2 = 3 + 2 = 5 < \sqrt{29}$ the circles do **NOT** touch.

16. Circle R has equation $x^2 + y^2 - 2x - 4y - 4 = 0$ and

Circle S has equation $(x-4)^2 + y(y-6)^2 = 4$

Show that circles R and S touch externally.

$2g = -2 \quad 2f = -4 \quad c = -4$

$g = -1 \quad f = -2$

so centre is $C_R(1, 2)$

Radius $= \sqrt{g^2 + f^2 - c}$

Radius $= \sqrt{(-1)^2 + (-2)^2 + 4}$

Radius $= \sqrt{9} = 3$

Centre is $C_S(4, 6)$

Radius $= \sqrt{4}$

Radius $= 2$

Centres $= \sqrt{(x_S - x_R)^2 + (y_S - y_R)^2}$

Centres $= \sqrt{(4-1)^2 + (6-2)^2}$

Centres $= \sqrt{3^2 + 4^2}$

Centres $= \sqrt{9 + 16} = \sqrt{25} = 5$

Since $R_1 + R_2 = 3 + 2 = 5$ the circles touch **EXTERNALLY**.