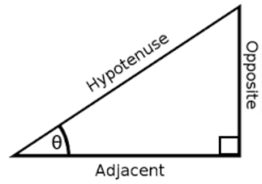
# Compound Angle

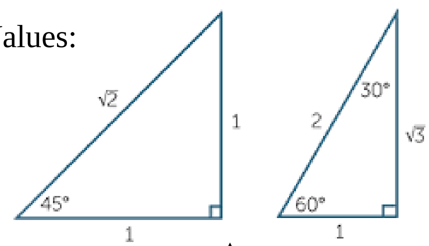
SPTA Mathematics – Higher Notes

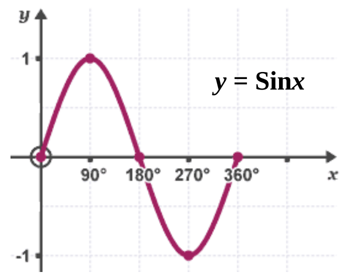
## Trig Stuff from Nat 5 & earlier in the Higher:

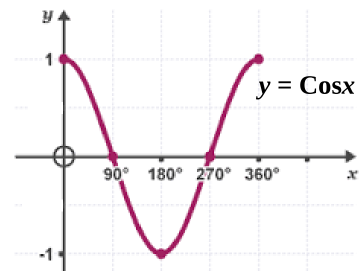



- SOHCAHTOA: $\sin\theta = \frac{opp}{hyp}$ $\cos\theta = \frac{adj}{hyp}$ $\tan\theta = \frac{opp}{adj}$

- Exact Values:

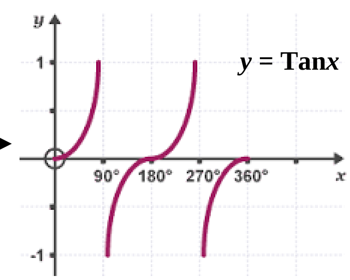



**Use the TRIANGLES for exact values of 30, 45 & 60**

**Use the GRAPHS for exact values of 0, 90, 180, 270 & 360**

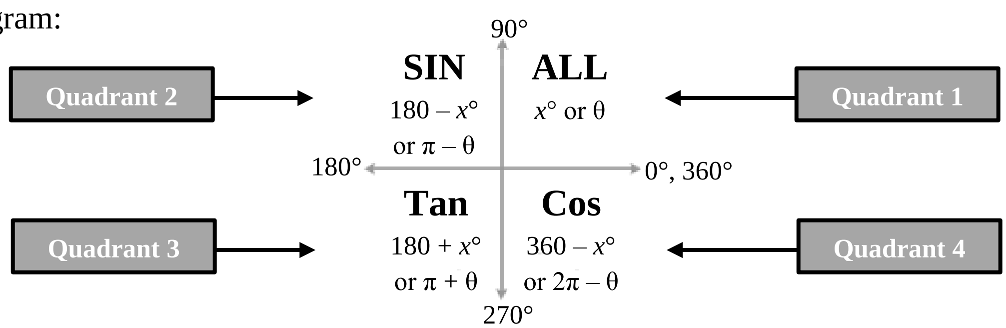



- Radians: $\pi$ radians = 180°

  Convert to RADIANS by dividing by 180 and multiplying by $\pi$

  Convert to DEGREES by dividing by $\pi$ and multiplying by 180

- CAST Diagram:




- Solving Trig Equations: You should already be able to solve Trig Equations in the form

$$y = a\sin(bx + c) + d \quad \text{or} \quad y = a\cos(bx + c) + d$$

- Trig Identities: $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and $\sin^2 x + \cos^2 x = 1$

# Compund Angle Formulae:

- Also known as the **ADDITION FORMULAE**.
- There are 4 Compound Angle Formulae as follows:

**1.** $\sin(A+B) = \sin A \cos B + \cos A \sin B$

**2.** $\sin(A-B) = \sin A \cos B - \cos A \sin B$

**3.** $\cos(A+B) = \cos A \cos B - \sin A \sin B$

**4.** $\cos(A-B) = \cos A \cos B + \sin A \sin B$

**How they appear on the SQA Formula List**

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

**Careful!! The signs are flipped!!!**

# Examples:

**1.** Expand the following:

**a)** $\cos(X+Y) = \cos X \cos Y - \sin X \sin Y$

**b)** $\sin(P-Q) = \sin P \cos Q - \cos P \sin Q$

**2.** Expand and simplify the following:

**a)** $\sin(180+y) = \sin 180 \cos y - \cos 180 \sin y$

$= (0)\cos y - (-1)\sin y$

$= \sin y$

**b)** $\cos\left(x - {}^{\pi}/_{2}\right) = \cos x \cos {}^{\pi}/_{2} + \sin x \sin {}^{\pi}/_{2}$

$= (0)\cos x + (1)\sin x$

$= \sin x$

**c)** $\cos(-p) = \cos(0-p)$

$= \cos 0 \cos p + \sin 0 \sin p$

$= (1)\cos p + (0)\sin p$

$= \cos p$

**Use the GRAPHS to find the exact values of 0, 90 & 180**

**Note: In part (b) above you can change the radians to degrees if you find this easier.**

**3.** Simplify the following:

**a)** $\cos 130 \cos 50 - \sin 130 \sin 50 = \cos(130 + 50)$

$= \cos(180)$

$= -1$

**b)** $\sin 5y \cos 2y - \cos 5y \sin 2y = \sin(5y - 2y)$

$= \sin(3y)$

**4.** Prove that: $\sin(60 + y) + \cos(y - 150) = \sin y$

$\sin(60 + y) + \cos(y + 150) = \sin 60 \cos y + \cos 60 \sin y + \cos y \cos 150 + \sin y \sin 150$

$= {}^{\sqrt{3}}/_{2} \cos y + {}^{1}/_{2} \sin y + \cos y \, (-{}^{\sqrt{3}}/_{2}) + \sin y \, ({}^{1}/_{2})$

$= {}^{\sqrt{3}}/_{2} \cos y + {}^{1}/_{2} \sin y - {}^{\sqrt{3}}/_{2} \cos y + {}^{1}/_{2} \sin y$

$= {}^{\sqrt{3}}/_{2} \cos y - {}^{\sqrt{3}}/_{2} \cos y + {}^{1}/_{2} \sin y + {}^{1}/_{2} \sin y$

$= \sin y$ as required.

**Use the** exact value triangles and the CAST diagram to find the **exact values of 60 & 150**

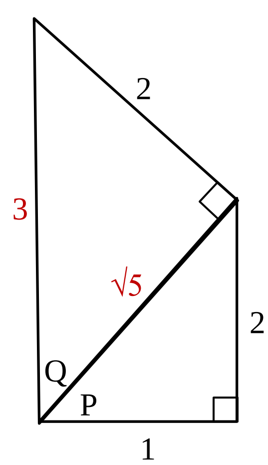
**5.** Show that $\sin(A + B) + \sin(A - B) = 2 \sin A \cos B$

$\sin(A + B) + \sin(A - B) = \sin A \cos B + \cos A \sin B + \sin A \cos B - \cos A \sin B$

$= 2 \sin A \cos B$ as required

**6.** Calculate the exact value of sin(*P* + *Q*)**:**

**Use Pythagoras to find the missing sides in the triangles, then use SOHCAHTOA**




$\sin P = \frac{2}{\sqrt{5}}$ $\sin Q = \frac{2}{3}$

$\cos P = \frac{1}{\sqrt{5}}$ $\cos Q = \frac{\sqrt{5}}{3}$

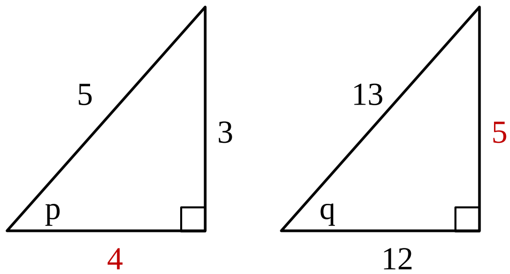
$\sin(P + Q) = \sin P \cos Q + \cos P \sin Q$

$= \frac{2}{\sqrt{5}} \times \frac{\sqrt{5}}{3} + \frac{1}{\sqrt{5}} \times \frac{2}{3}$

$= \frac{2\sqrt{5}}{3\sqrt{5}} + \frac{2}{3\sqrt{5}}$

$= \frac{2\sqrt{5}+2}{3\sqrt{5}}$

**7.** Given that $\sin p = \frac{3}{5}$ and $\cos q = \frac{12}{13}$ prove that $\cos(p-q) = \frac{63}{65}$




$\cos p = \frac{4}{5}$

$\sin q = \frac{5}{13}$

**Use Pythagoras to find the missing sides in the triangles, then use SOHCAHTOA**

$\cos(p-q) = \cos p \cos q + \sin p \sin q$

$= \frac{4}{5} \times \frac{12}{13} + \frac{3}{5} \times \frac{5}{13}$

$= \frac{48}{65} + \frac{15}{65}$

$= \frac{63}{65}$ as required.

**8.** By expressing 75° = 45° + 30° find the exact value of sin75°

$\sin 75 = \mathrm{Sin}(45+30) \ = \sin 45 \cos 30 + \cos 45 \sin 30$

$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2}$

$= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}$

$= \frac{\sqrt{3}+1}{2\sqrt{2}}$

**Use the** exact value triangles to find the **exact values of 30 & 45**

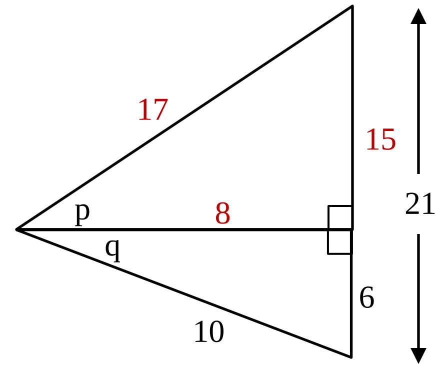
**This type of question may ask you to express your answer with a RATIONAL DENOMINATOR. This is from National 5 and only needs to be done if stated in the question. It is done as follows:**

$= \frac{\sqrt{3}+1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{\sqrt{6}+\sqrt{2}}{2\sqrt{4}}$

$= \frac{\sqrt{6}+\sqrt{2}}{4}$

**9.** Using the diagram opposite show that:




**a)** **(i)** $\sin(p+q) = \frac{84}{85}$ **(ii)** $\cos(p+q) = -\frac{13}{85}$

**b)** Hence find the exact value of $\tan(p+q)$

**a)** **(i)** $\sin p = \frac{15}{17}$ $\cos p = \frac{8}{17}$

$\sin q = \frac{6}{10} = \frac{3}{5}$ $\cos q = \frac{8}{10} = \frac{4}{5}$

$$\sin(p+q) = \sin p \cos q + \cos p \sin q$$

$$= \frac{15}{17} \times \frac{4}{5} + \frac{8}{17} \times \frac{3}{5}$$

$$= \frac{60}{85} + \frac{24}{85}$$

$= \frac{84}{85}$ as required.

**(ii)** $\cos(p+q) = \cos p \cos q - \sin p \sin q$

$$= \frac{8}{17} \times \frac{4}{5} - \frac{15}{17} \times \frac{3}{5}$$

$$= \frac{32}{85} - \frac{45}{85}$$

$= -\frac{13}{85}$ as required.

**b)** $\tan\theta = \frac{\sin\theta}{\cos\theta}$ so $\tan(p+q) = \frac{\sin(p+q)}{\cos(p+q)}$

$$= \frac{84}{85} \div \frac{-13}{85}$$

$$= \frac{84}{85} \times \frac{85}{-13}$$

$$= \frac{84}{-13}$$

$= -\frac{84}{13}$ or $-6\frac{6}{13}$

**Unless stated in the question you can leave your answer as either a Top Heavy Fraction or a Mixed Number.**

# Double Angle Formulae:

- There are also 4 Double Angle Formulae as follows:

  **1.** $\sin 2A = 2\sin A\cos A$

  **2.** $\cos 2A = \cos^2 A - \sin^2 A$

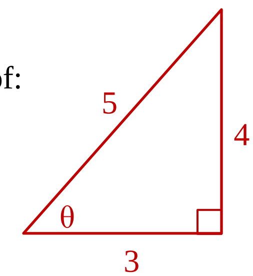
  **3.** $\cos 2A = 2\cos^2 A - 1$

  **4.** $\cos 2A = 1 - 2\sin^2 A$

**ALL 4 are given on the SQA Formula List**

# Examples:

**10.** Given that $\tan\theta = \frac{4}{3}$ calculate the exact value of:




$\sin\theta = \frac{4}{5}$

$\cos\theta = \frac{3}{5}$

**a)** $\sin 2\theta$: $\sin 2\theta = 2\sin\theta\cos\theta$

$= 2 \times \frac{4}{5} \times \frac{3}{5}$

$= \frac{24}{25}$

**b)** $\cos 2\theta$: $\cos 2\theta = 2\cos^2\theta - 1$

$= 2 \times \left(\frac{3}{5}\right)^2 - 1$

$= 2 \times \frac{9}{25} - 1$

$= \frac{18}{25} - 1$

$= \frac{18}{25} - \frac{25}{25} = -\frac{7}{25}$

**You can use any of the 3 formulae to get the same answer!!**

$\cos 2\theta = 1 - 2\sin^2\theta$

$= 1 - 2 \times \left(\frac{4}{5}\right)^2$

$= 1 - 2 \times \frac{16}{25}$

$= 1 - \frac{32}{25}$

$= \frac{25}{25} - \frac{32}{25} = -\frac{7}{25}$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$= \left(\frac{3}{5}\right)^2 - \left(\frac{4}{5}\right)^2$

$= \frac{9}{25} - \frac{16}{25} = -\frac{7}{25}$

**c)** $\tan 2\theta$: $\tan\theta = \frac{\sin\theta}{\cos\theta}$ so $\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta}$

$= \frac{24}{25} \div \frac{-7}{25}$

$= \frac{24}{25} \times \frac{25}{-7}$

$= \frac{24}{-7}$ or $-2\frac{3}{7}$

**d)** cos4θ: $\cos 4\theta = \cos^2 2\theta - \sin^2 2\theta$

$$= \left(-\frac{7}{25}\right)^2 - \left(\frac{24}{25}\right)^2$$

$$= \frac{49}{625} - \frac{576}{625} = -\frac{527}{625}$$

**Use your answers from part (a) & (b)**

**When asked to find cos2θ you can use <u>ANY</u> of the 3 formulae expressed above and always get the same answer!!**

**11.** Given that $\cos\theta = \frac{2}{\sqrt{3}}$ calculate the exact value of cos2θ

$$\cos 2\theta = 2\cos^2\theta - 1$$

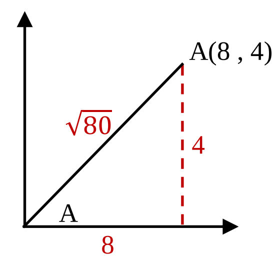
$$= 2 \times \left(\frac{2}{\sqrt{3}}\right)^2 - 1$$

$$= 2 \times \frac{4}{3} - 1$$

$$= \frac{8}{3} - 1$$

$$= \frac{8}{3} - \frac{3}{3} = \frac{5}{3}$$

**12.** For the diagram shown find the exact values of:




$$\sin A = \frac{4}{\sqrt{80}} = \frac{4}{4\sqrt{5}} = \frac{1}{\sqrt{5}}$$

$$\cos A = \frac{8}{\sqrt{80}} = \frac{2}{\sqrt{5}}$$

**a)** sin2A: $\sin 2A = 2 \sin A \cos A$

$$= 2 \times \frac{1}{\sqrt{5}} \times \frac{2}{\sqrt{5}}$$

$$= \frac{4}{5}$$

**b)** cos2A: $\cos 2A = 1 - 2\sin^2 A$

$$= 1 - 2 \times \left(\frac{1}{\sqrt{5}}\right)^2$$

$$= 1 - 2 \times \frac{1}{5}$$

$$= 1 - \frac{2}{5} = \frac{3}{5}$$

**c)** By expressing $\cos 3A$ as $\cos(2A + A)$ find the exact value of $\cos 3A$ expressing your answer with a rational denominator:

$$\cos 3A = \cos(2A + A) = \cos 2A \cos A - \sin 2A \sin A$$

$$= \frac{3}{5} \times \frac{2}{\sqrt{5}} - \frac{4}{5} \times \frac{1}{\sqrt{5}}$$

$$= \frac{6}{5\sqrt{5}} - \frac{4}{5\sqrt{5}}$$

$$= \frac{2}{5\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$$

$$= \frac{2\sqrt{5}}{5 \times 5} = \frac{2\sqrt{5}}{25}$$

**12.** Show that $\sin 3x = 3\sin x - 4\sin^3 x$:

$$\sin 3x = \sin(2x + x) = \sin 2x\cos x + \cos 2x\sin x$$

**Expand sin2*x* & cos2*x*** →

$$= (2\sin x\cos x)\cos x + (1 - 2\sin^2 x)\sin x$$

$$= 2\sin x\cos^2 x + \sin x - 2\sin^3 x$$

**Rearrange $\cos^2 x + \sin^2 x = 1$ to replace $\cos^2 x$** →

$$= 2\sin x(1 - \sin^2 x) + \sin x - 2\sin^3 x$$

$$= 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x$$

$$= 3\sin x - 4\sin^3 x \quad \text{as required}$$

# Solving Trig Equations:

- The Double Angle Formulae along with Factorisation can be used to solve equations in the forms:

**1.** $a\sin 2x + b\sin x = 0$ or $a\sin 2x + b\cos x = 0$ — by expanding sin2*x* and then taking out a common factor.

**2.** $y = a\cos 2x + b\sin x + c$ or $y = a\cos 2x + b\cos x + c$ — by expanding cos2*x* and then using Quadratic Factorisation.

**Don't use $\cos^2 x - \sin^2 x$ to replace cos2*x***

# Examples:

**13.** Solve $2\sin 2x + 3\sin x = 0$ where $0 \le x \le 180°$

$$2\sin 2x + 3\sin x = 0$$

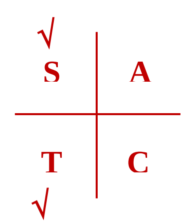
$$2(2\sin x\cos x) + 3\sin x = 0$$

$$4\sin x\cos x + 3\sin x = 0$$

$$\sin x(4\cos x + 3) = 0$$

$\sin x = 0$ or $4\cos x + 3 = 0$

$x = 0°, 180°, \cancel{360°}$ or $\cos x = -\frac{3}{4}$




$$x = 180 - 41.41°, 180 + 41.41°$$

$$x = 138.59°, \cancel{221.41°}$$

**Always check the range!!**

$$x = 0°, 138.59° \,\&\, 180°$$

**14.** Solve $3\cos 2\theta + 10\cos\theta = 1$ where $0 \leq x \leq \pi$

$$3\cos 2\theta + 10\cos\theta - 1 = 0$$
$$3(2\cos^2\theta - 1) + 10\cos\theta - 1 = 0$$
$$6\cos^2\theta - 3 + 10\cos\theta - 1 = 0$$
$$6\cos^2\theta + 10\cos\theta - 4 = 0$$
$$2(3\cos^2\theta + 5\cos\theta - 2) = 0$$
$$2(3\cos\theta - 1)(\cos\theta + 2) = 0$$

**Consider:** $3x^2 + 5x - 2$
$(3x - 1)(x + 2)$

$3\cos\theta - 1 = 0$ *or* $\cos\theta + 2 = 0$

$\cos\theta = \frac{1}{3}$ *or* ~~$\cos\theta = -2$~~ ← **Not possible!!**

$\theta = 70.53°, 360 - 70.53°$

$\theta = 70.53°,$ ~~$289.47°$~~ ← **Not in the range!!**

$\theta = 70.53°$

$\theta = 1.23\ radians$ ← **÷ 180 x π**

| S | A √ |
|---|---|
| T | C √ |

**15.** Solve $\cos 2x + 2\sin x = \sin^2 x$ where $0° \leq x \leq 360°$

$$\cos 2x + 2\sin x - \sin^2 x = 0$$
$$1 - 2\sin^2 x + 2\sin x - \sin^2 x = 0$$
$$1 + 2\sin x - 3\sin^2 x = 0$$
$$-3\sin^2 x + 2\sin x + 1 = 0$$
$$-1(3\sin^2 x - 2\sin x - 1) = 0$$
$$-1(3\sin x + 1)(\sin x - 1) = 0$$

**Consider:** $3x^2 - 2x - 1$
$(3x + 1)(x - 1)$

$3\sin x + 1 = 0$ *or* $\sin x - 1 = 0$

$\sin x = -\frac{1}{3}$ *or* $\sin x = 1$

| S | A √ |
|---|---|
| T | C √ |

$x = 180 + 19.47°, 360 - 19.47°$ $\qquad x = 90°$

$x = 199.47°, 340.53°$

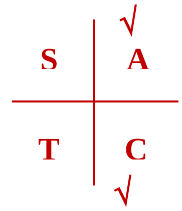
$$x = 90°, 199.47°\ \&\ 340.53°$$

**16.** Solve $\sin\theta = \sin 2\theta$ where $0 \le \theta \le 2\pi$

$$\sin\theta - \sin 2\theta = 0$$

$$\sin\theta - 2\sin\theta\cos\theta = 0$$

$$\sin\theta(1 - 2\cos\theta) = 0$$




$sin\theta = 0$ or $1 - 2\cos\theta = 0$

$\theta = 0°, 180°, 360°$ or $cos\theta = \frac{1}{2}$

$\theta = 60°, 360 - 60°$

$$\theta = 0°, 60°, 180°, 300° \ \& \ 360°$$

$$\theta = 0, \frac{\pi}{3}, \pi, \frac{5\pi}{3} \ \& \ 2\pi$$

**17. a)** Express the function, $f(a) = 6\sin^2 a - \cos a$ , in the form $f(a) = p\cos^2 a + q\cos a + r$ and write down the values of $p$, $q$ and $r$.

$$f(a) = 6\sin^2 a - \cos a$$

$$f(a) = 6(1 - \cos^2 a) - \cos a$$

$$f(a) = 6 - 6\cos^2 a - \cos a$$

$$f(a) = -6\cos^2 a - \cos a + 6$$

$p = -6, \ q = -1 \ \& \ r = 6$

**b)** Hence, or otherwise, solve $6\sin^2 a - \cos a = 5$ where $0° \le a \le 360°$

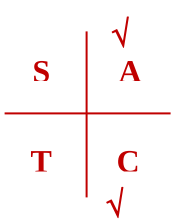
$$6\sin^2 a - \cos a = 5$$

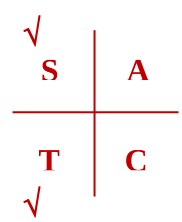
$$-6\cos^2 a - \cos a + 6 = 5$$

$$-6\cos^2 a - \cos a + 1 = 0$$

$$-1(6\cos^2 a + \cos a - 1) = 0$$

$$-1(3\cos a - 1)(2\cos a + 1) = 0$$




$\cos a = \frac{1}{3}$ or $\cos a = -\frac{1}{2}$

$a = 70.53°, 360 - 70.53°$ or $a = 180 - 60°, 180 + 60°$

$a = 70.53°, 289.47°$ or $a = 120°, 240°$

$$a = 70.53°, 120°, 240° \ \& \ 289.47°$$