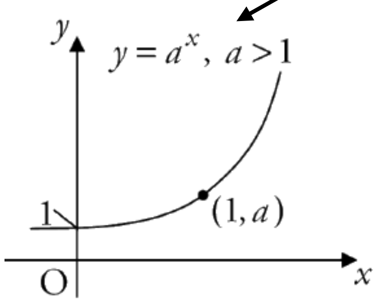
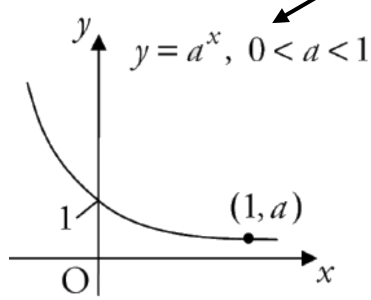
Exponential – We have seen most of this already:

- An exponential equation (function) to the base a is in the form: $y = a^x$ (or $f(x) = a^x$), $a > 0$
- When $x = 0$ then $y = a^0 = 1$ and when $x = 1$ then $y = a^1 = a$ hence the graph of an exponential will always pass through the 2 points: $(0 , 1)$ and $(1 , a)$. ← **Must learn**
- If $a > 1$ then the graph is called a GROWTH curve and when $0 < a < 1$ it is called a DECAY curve:



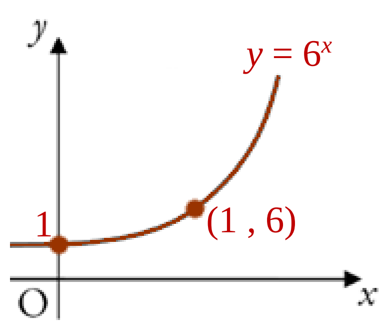


- The graph gets very close to the x-axis, but never touches it. The x-axis is said to be an ASYMPTOTE to the curve.
- If we know 2 points on the graph we can find the equation of the curve.
- $f(x) = e^x$ is called the exponential function to the base e. It is sometimes written as $exp(x)$.
- Your calculator will have an e^x button, usually above the *ln* button(see logarithms below).
- The constant $e = 2.718281828...$ is an important number in Maths and like π it is irrational.

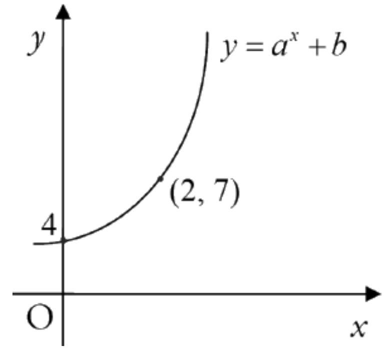
Examples:

1. Sketch the curve $y = 6^x$

Since $a = 6$ the graph passes through the points: $(0 , 1)$ and $(1 , 6)$



2. For the graph below find the values of a and b and then state the curves equation.



Use (0 , 4): $4 = a^0 + b \Longrightarrow 4 = 1 + b \Longrightarrow b = 3$

Use (2 , 7): $7 = a^2 + 3 \Longrightarrow 4 = a^2 \Longrightarrow a = \sqrt{4}$

$\Longrightarrow a = 2$

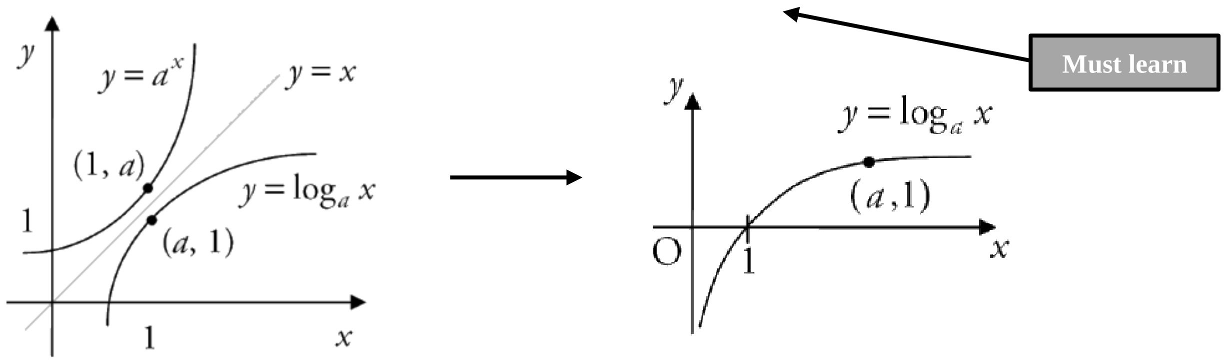
So $y = 2^x + 3$

3. Find $f(x) = e^x$, when $x = 5$:

$f(5) = e^5 = 148.41$

Logarithms – We have seen most of this already:

- A Logarithmic equation(function), shortened to *Log*, is the inverse of an exponential function.
- A *Log* equation (function) to the base a is written in the form: $y = Log_a x$ (or $f(x) = Log_a x$), $a > 0$.
- As it is the inverse of exponentials its graph is the refelection of a GROWTH curve in the line $y = x$.
- Hence a *Log* curve will always pass through the points: (1 , 0) and (a , 1) as shown below:

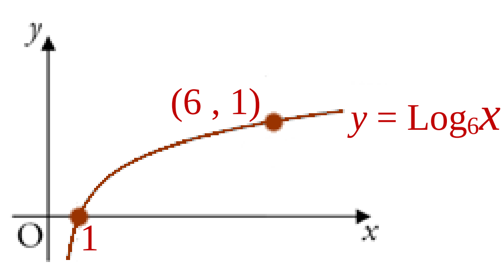


- The graph gets very close to the y-axis, but never touches it. The y-axis is said to be an ASYMPTOTE to the curve.
- In general: $Log_a 1 = 0$ (*Log* to any base of 1 is zero) and
  $Log_a a = 1$ (*Log* to base a of a is one).
- If we know 2 points on the graph we can find the equation of the curve.
- The *Log* key on your calculator is to the base 10, i.e. Log_{10}
- Log_e, usually written as *ln*, is called the NATURAL LOGARITHM and is also on the calculator.

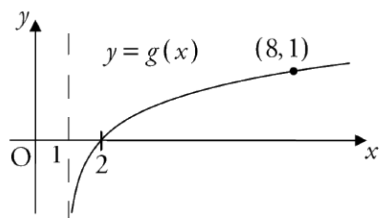
Examples:

4. Sketch the curve $y = Log_6 x$

Since $a = 6$ the graph passes through the points: (1 , 0) and (6 , 1)



5. A graph of the function $g(x) = Log_a(x + b)$, where a, b are constants and $a > 1$ is shown here:



Find the values of a and b and state $g(x)$.

Curve normally passes through (1 , 0), but is passing through (2 , 0), so has been shifted right 1, giving $b = -1$.

Use (8 , 1): $1 = Log_a(8 - 1) \implies Log_a 7 = 1 \implies a = 7$

So $g(x) = Log_7 x$

6. Calculate:

a) $Log_{10}75 = 1.875$ **(b)** $ln75 = 4.3175$

Exponentials ↔ Logarithms:

- $y = Log_a x \leftrightarrow x = a^y$, where a , $x > 0$ ← **Must learn**

Examples:

7. Express each of the following in exponential form and solve if possible:

a) $Log_5 125 = 3 \implies 125 = 5^3$

b) $Log_3 x = y \implies x = 3^y$

c) $Log_2 x = 3 \implies x = 2^3 \implies x = 8$

d) $Log_3 (2x - 5) = 2 \implies 2x - 5 = 3^2$
$\implies 2x - 5 = 9$
$\implies 2x = 14$
$\implies x = 7$

8. Express each of the following in Logarithmic form:

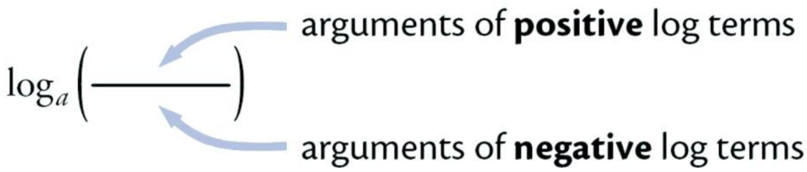
a) $2^3 = 8 \implies Log_2 8 = 3$

b) $729 = 3^6 \implies Log_3 729 = 6$

c) $x = 8^y \implies Log_8 x = y$

Laws of Logarithms:

- Remember that $Log_a\ a = 1$ and $Log_a\ 1 = 0$ ← **Must learn**
- There are 3 Laws of Logarithms that you must remember, they are **NOT** on the Formulae Sheet:
    1. $Log_a\ x + Log_a\ y = Log_a\ xy$, where a , x , y >0
    2. $Log_a\ x - Log_a\ y = Log_a\ {}^{x}/_{y}$, where a , x , y >0 ← **Must learn**
    3. $Log_a\ x^n = nLog_a\ x$, where a , x >0
- For Laws 1 & 2 the bases of each *Log* must be same.
- We can use the Laws of Logarithms to solve equations involving Logs.
- When there are several additions or subtractions of Logs they can be combined into one as follows:



Examples:

9. Solve the following:

a) $Log_3\ 1 + Log_3\ 9$

$= Log_3\ (1 \times 9)$ ← **Rule 1**

$= Log_3\ 9$

$= Log_3\ 3^2$

$= 2\ Log_3\ 3$ ← **Rule 3**

$= 2 \times 1$

$= 2$

(b) $Log_4\ 6 - Log_4\ 3$

$= Log_4\ {}^{6}/_{3}$ ← **Rule 2**

$= Log_4\ 2$

$= Log_4\ 4^{1/2}$

$= {}^{1}/_{2}\ Log_4\ 4$ ← **Rule 3**

$= {}^{1}/_{2} \times 1$

$= {}^{1}/_{2}$

c) $Log_6\ 4 + 2\ Log_6\ 3$

$= Log_6\ 4 + Log_6\ 3^2$ ← **Rule 3**

$= Log_6\ 4 + Log_6\ 9$

$= Log_6\ (4 \times 9)$ ← **Rule 1**

$= Log_6\ 36$

$= Log_6\ 6^2$

$= 2\ Log_6\ 6$ ← **Rule 3**

$= 2 \times 1$

$= 2$

(d) $Log_2\ 3 - Log_2\ 6 + Log_2\ 2 - Log_2\ 8$

$= Log_2\ ({}^{3 \times 2}/_{6 \times 8})$ ← **Rule 1 & 2**

$= Log_2\ ({}^{6}/_{48})$

$= Log_2\ ({}^{1}/_{8})$

$= Log_2\ ({}^{1}/_{2^3})$

$= Log_2\ 2^{-3}$

$= -3\ Log_2\ 2$ ← **Rule 3**

$= -3 \times 1$

$= -3$

10. Simplify $4Log_e(2e) - 3Log_e(3e)$, expressing your answer in the form $a + Log_e\, b - Log_e\, c$, where a , b and c are whole numbers.

$4Log_e(2e) - 3Log_e(3e)$

$= Log_e(2e)^4 - Log_e(3e)^3$

$= Log_e(16e^4) - Log_e(27e^3)$

$= Log_e\left(\frac{16e^4}{27e^3}\right)$

$= Log_e\left(\frac{16e}{27}\right)$

$= Log_e(16e) - Log_e 27$

$= Log_e 16 + Log_e e - Log_e 27$

$= 1 + Log_e 16 - Log_e 27$

OR

$4Log_e(2e) - 3Log_e(3e)$

$= 4(Log_e 2 + Log_e e) - 3(Log_e 3 + Log_e e)$

$= 4Log_e 2 + 4Log_e e - 3Log_e 3 - 3Log_e e$

$= Log_e 2^4 + Log_e e - Log_e 3^3$

$= Log_e 16 + 1 - Log_e 27$

$= 1 + Log_e 16 - Log_e 27$

11. Solve:

a) $Log_a 13 + Log_a x = Log_a 273$

$Log_a 13x = Log_a 273$

$13x = 273$

$x = 21$

Cancel Logs on both sides.

(b) $Log_2 7 = Log_2 x + 3$

$Log_2 7 - Log_2 x = 3$

$Log_2\left(\frac{7}{x}\right) = 3$

$^7/_x = 2^3$

$7 = 8x$

$x = {}^7/_8$

Convert Logs to exponential.

c) $Log_{11}(4x+3) - Log_{11}(2x-3) = 1$

$Log_{11}\left(\frac{4x+3}{2x-3}\right) = 1$

$\frac{4x+3}{2x-3} = 11^1$

$4x + 3 = 11(2x - 3)$

$4x + 3 = 22x - 33$

$-18x + 3 = -33$

$-18x = -36$

$x = 2$

(d) $e^x = 7$

$\ln e^x = \ln 7$

$x \ln e = \ln 7$

$x = \ln 7$

$x = 1.9$

Take Logs of both sides.

12. Solve $Log_a(2p + 1) + Log_a(3p - 10) = Log_a(11p)$ for $p > 4$

$$Log_a(2p + 1) + Log_a(3p - 10) = Log_a(11p)$$

$$Log_a[(2p + 1)(3p - 10)] = Log_a(11p)$$

Cancel Logs on both sides.

$$(2p + 1)(3p - 10) = 11p$$

$$6p^2 - 17p - 10 = 11p$$

$$6p^2 - 17p - 10 - 11p = 0$$

$$6p^2 - 28p - 10 = 0$$

$$(3p + 1)(p - 5) = 0$$

$$3p + 1 = 0 \quad \text{or} \quad p - 5 = 0$$

$$p = -{}^1/_3, 5 \quad \text{so } p = 5 \text{ since } p > 4$$

13. For the formula the speed, s, of an object at time, t minutes is given by: $s(t) = 100e^{3t}$

a) Evaluate the initial speed.

$$s(0) = 100e^{3(0)}$$

$$= 100e^0$$

$$= 100$$

(b) Calculate the time taken for the initial speed to be trebled.

$$s(t) = 100e^{3t}$$

$$300 = 100e^{3t}$$

$$e^{3t} = 3$$

$$ln\ e^{3t} = ln\ 3$$

Take Logs of both sides.

$$3t\ ln\ e = 1.099$$

$$3t = 1.099$$

$$t = 0.366 \text{ minutes}$$

14. A radioactive substance has Mass, M grams, at t years such that M is given by the formula:
$M_t = M_0 e^{-0.03t}$ where M_0 is the initial mass in grams.

a) If M_0 = 800g find the mass after 7 years.

$$M_7 = 800\, e^{-0.03 \times 7}$$

$$= 800e^{-0.21}$$

$$= 648.47\text{g}$$

(b) Calculate the half life of this substance.

$$400 = 800\, e^{-0.03t}$$

$$0.5 = e^{-0.03t}$$

$$\ln e^{-0.03t} = ln\ 0.5$$

Take Logs of both sides.

$$-0.03t\ ln\ e = -0.69315$$

$$-0.03t = -0.69315$$

$$t = 23.105 \text{ years}$$

Experimental Data:

- Often Science experiments result in graphs of the forms: $y = ax^b$ or $y = ab^x$
- It is often more useful to have this information in the form of a straight line as it is then easier to perform calculations on. We can transform these equations into the form $Y = mX + C$ using Logs.
- Type 1:

$$y = ax^b$$
$$Log\,y = Log(ax^b)$$
$$Log\,y = Log\,a + Log\,x^b$$
$$Log\,y = Log\,a + b\,Log\,x$$
$$Log\,y = b\,Log\,x + Log\,a$$
$$Y = m\,X + C$$

Base of Logs doesn't matter.

Type 2:

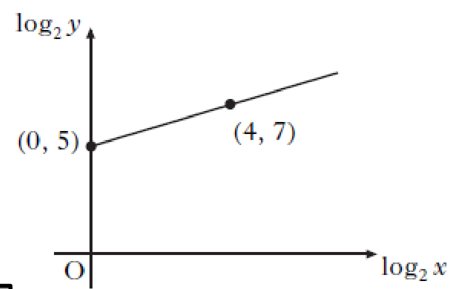
$$y = ab^x$$
$$Log\,y = Log(ab^x)$$
$$Log\,y = Log\,a + Log\,b^x$$
$$Log\,y = Log\,a + x\,Log\,b$$
$$Log\,y = (Log\,b)\,x + Log\,a$$
$$Y = \mathrm{M}\,x + C$$

Examples:

15. Type 1: Variables x and y are related by the equation $y = kx^n$

The graph of $Log_2\,y$ against $Log_2\,x$ is a straight line through (0 , 5) and (4 , 7) as shown in the diagram.

Find the values of k and n.



$$y = kx^n$$
$$Log_2\,y = Log_2\,(kx^n)$$
$$Log_2\,y = Log_2\,k + Log_2\,x^n$$
$$Log_2\,y = Log_2\,k + n\,Log_2\,x$$
$$Log_2\,y = n\,Log_2\,x + Log_2\,k$$

So $Y = mX + C$

Use coordinates to find Gradient.

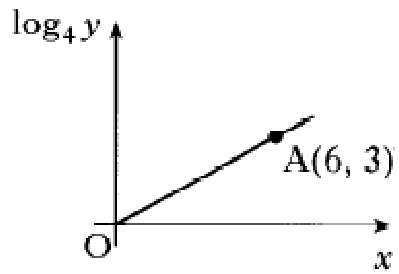
$$m = \frac{y_B - y_A}{x_B - x_A}$$
$$n = \frac{7-5}{4-0}$$
$$= \frac{2}{4} = \frac{1}{2}$$

y-intercept

$$C = 5$$
$$Log_2\,k = 5$$
$$k = 2^5$$
$$k = 32$$

So $y = 32x^{1/2}$

16. Type 2: Two variables x and y are connected by the law $y = ab^x$. The graph of $Log_4 y$ against x is a straight line through the origin and (6 , 3).



State the equation in the form: $y = ab^x$

$y = ab^x$

$Log_4 y = Log_4 a + Log_4 b^x$

$Log_4 y = Log_4 a + x Log_4 b$

$Log_4 y = (Log_4 b)x + Log_4 a$

$Y = Mx + C$

$m = \frac{y_B - y_A}{x_B - x_A}$

$= \frac{3-0}{6-0}$

$= \frac{3}{6} = \frac{1}{2}$

$Log_4 a = 0$

$a = 4^0$

$a = 1$

$Log_4 b = ½$

$b = 4^{1/2}$

$b = \sqrt{4}$

$b = 2$

So $y = 2^x$