

H

Integration

SPTA Mathematics - Higher Notes



- Integration is the reverse of Differentiation and is sometimes known as the Anti-Derivative.
- The symbol, \int , is used to represent Integration and is always used in conjunction with dx meaning "with respect to x "
- Must include!**
 $\int ax^n dx = \frac{ax^{n+1}}{n+1} + c$ where c is the Constant of Integration.
- Integrals that include the Constant of Integration are known as **INDEFINITE INTEGRALS**.
- The Integral of a constant is x .
- Prior to Integrating a function the following must be true:
 - All brackets should be multiplied out.
 - Roots need to be changed to powers: $\sqrt[m]{x^n} = x^{n/m}$
 - x cannot appear in the denominator of a fraction: $\frac{a}{bx^n} = \frac{a}{b}x^{-n}$
- After Integrating it is good practice to return the expression to the form the question gave it in.

Examples:

1. Integrate the following:

Should always match
the variable being
Integrated

$$\begin{aligned} \text{a) } \int 10x^4 dx &= \frac{10x^5}{5} \\ &= 2x^5 + c \end{aligned}$$

$$\begin{aligned} \text{b) } \int 5m^{-2} dm &= \frac{5m^{-1}}{-1} \\ &= -5m^{-1} \\ &= -\frac{5}{m} + c \end{aligned}$$

$$\begin{aligned} \text{c) } \int 7 dx &= 7x + c \end{aligned}$$

$$\begin{aligned} \text{d) } \int x^{3/2} dx &= \frac{x^{5/2}}{5/2} \\ &= \frac{2x^{5/2}}{5} + c \end{aligned}$$

$$\begin{aligned} \text{e) } \int \sqrt[3]{a^4} da &= \int a^{4/3} da \\ &= \frac{a^{7/3}}{7/3} \\ &= \frac{3\sqrt[3]{a^7}}{7} + c \end{aligned}$$

$$\begin{aligned} \text{f) } \int \frac{2}{3x^4} dx &= \int \frac{2}{3} x^{-4} dx \\ &= \frac{2}{3} \times \frac{x^{-3}}{-3} \\ &= \frac{2x^{-3}}{-9} = \frac{2}{-9x^3} + c \end{aligned}$$

2. Integrate the following:

a) $\int 3x^4 - 4x^3 + 5 dx$

$$= \frac{3x^5}{5} - \frac{4x^4}{4} + 5x$$

$$= \frac{3x^5}{5} - x^4 + 5x + c$$

b) $\int \frac{dx}{5x^3}$

Alternative
format but
means the
same.

$$\int \frac{1}{5x^3} dx$$

$$\int \frac{1}{5} x^{-3} dx$$

$$= \frac{1}{5} \times \frac{x^{-2}}{-2}$$

$$= -\frac{x^{-2}}{10} = -\frac{1}{10x^2} + c$$

c) $\int \frac{2x^4 - 6x}{5} dx$

$$\int \frac{2x^4}{5} - \frac{6x}{5} dx$$

$$= \frac{2}{5} \times \frac{x^5}{5} - \frac{6}{5} \times \frac{x^2}{2}$$

$$= \frac{2x^5}{25} - \frac{6x^2}{10}$$

$$= \frac{2x^5}{25} - \frac{3x^2}{5} + c$$

d) $\int \frac{\sqrt{n+3}}{n^2} dn$

Split into
separate
fractions and
simplify before
integrating – do
NOT bring the
bottom up!!!!

$$\int \frac{n^{1/2}}{n^2} - \frac{3}{n^2} dn$$

$$\int n^{-3/2} - 3n^{-2} dn$$

$$= \frac{n^{-1/2}}{-1/2} - \frac{3n^{-1}}{-1}$$

$$= \frac{-2}{n^{1/2}} + \frac{3}{n}$$

$$= \frac{-2}{\sqrt{n}} + \frac{3}{n} + c$$

Definite Integrals:

- All the above examples were Indefinite Integrals as they contained the Constant of Integration, c .
- A definite Integral gives a specific solution based on 2 boundary values, called Limits.
- To find a Definite Integral we do the following
 - Find the Integral as normal but miss out the $+ c$
 - Sub in the Upper Limit and calculate the Integrals value.
 - Sub in the Lower Limit and calculate the Integrals value.
 - Subtract the 2 answers to find the Definite Integral value.
- This can be written as: If $F(x)$ is the integral of a function $f(x)$ then $\int_a^b f(x) dx = F(b) - F(a)$

Examples:

3. Evaluate these Definite Integrals:

a) $\int_1^5 4x^2 dx$

$$= \left[\frac{4x^3}{3} \right]_1^5$$

$$= \frac{4(5)^3}{3} - \left(\frac{4(1)^3}{3} \right)$$

$$= \frac{500}{3} - \left(\frac{4}{3} \right)$$

$$= \frac{496}{3}$$

Keep the 2nd limit
in brackets incase
it's negative

b) $\int_{-2}^0 x^3 + 3x^2 dx$

$$= \left[\frac{x^4}{4} + \frac{3x^3}{3} \right]_{-2}^0$$

$$= \left[\frac{x^4}{4} + 3x^3 \right]_{-2}^0$$

$$= \frac{0^4}{4} + 3(0)^3 - \left(\frac{(-2)^4}{4} + 3(-2)^3 \right)$$

$$= 0 - \left(\frac{16}{4} - 24 \right)$$

$$= 0 - (4 - 24)$$

$$= 0 - (-20)$$

$$= 20$$

c) $\int_{-1}^3 \frac{3}{x^5} dx$

$$\int_{-1}^3 3x^{-5} dx$$

$$= \left[\frac{3x^{-4}}{-4} \right]_{-1}^3$$

$$= \left[-\frac{3}{4x^4} \right]_{-1}^3$$

$$= -\frac{3}{4(3)^4} - \left(-\frac{3}{4(-1)^4} \right)$$

$$= -\frac{3}{324} - \left(-\frac{3}{4} \right)$$

$$= -\frac{1}{108} + \frac{81}{108}$$

$$= \frac{80}{108}$$

$$= \frac{20}{27}$$

$$= \frac{20}{27}$$

4. Find the value of a for $\int_1^a 1 + 2x dx = 4$

$$= \left[x + \frac{2x^2}{2} \right]_1^a$$

$$= [x + x^2]_1^a$$

$$= a + a^2 - (1 + 1^2)$$

$$= a^2 + a - 2$$

$$\text{So } a^2 + a - 2 = 4$$

$$a^2 + a - 6 = 0$$

$$(a + 3)(a - 2) = 0$$

$$a = \cancel{-3} \text{ or } a = 2$$

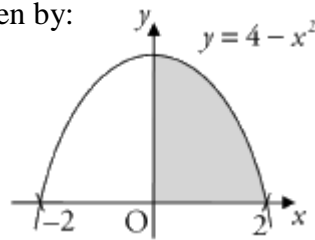
$$a = 2$$

Can only be 2 as it's
the upper limit and
the lower limit is 1.

Area Under a Curve:

- One interpretation of Definite Integrals is finding the Area under a Curve.
- It actually finds the Area enclosed by the Graph, between 2 Limits and the x -axis.
- So for the Graph opposite the Shaded Area would be given by:

$$\text{Area} = \int_0^2 4 - x^2 \, dx$$



Examples:

4. Find the shaded area above:

a) $\text{Area} = \int_0^2 4 - x^2 \, dx$

$$= \left[4x - \frac{x^3}{3} \right]_0^2$$

$$= 4(2) - \frac{(2)^3}{3} - \left(4(0) - \frac{(0)^3}{3} \right)$$

$$= 8 - \frac{8}{3} - (0)$$

$$= \frac{24}{3} - \frac{8}{3}$$

$$= \frac{16}{3} \text{ units}^2$$

b) Find the shaded area shown in this graph:

$$\text{Area} = \int_4^5 x^2 - 4x \, dx$$

$$= \left[\frac{x^3}{3} - \frac{4x^2}{2} \right]_4^5$$

$$= \left[\frac{x^3}{3} - 2x^2 \right]_4^5$$

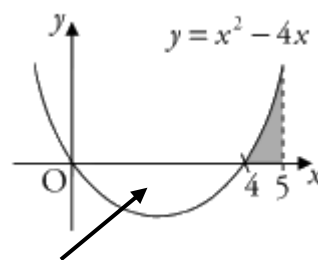
$$= \frac{(5)^3}{3} - 2(5)^2 - \left(\frac{(4)^3}{3} - 2(4)^2 \right)$$

$$= \frac{125}{3} - 50 - \left(\frac{64}{3} - 32 \right)$$

$$= \frac{61}{3} - 18$$

$$= \frac{61}{3} - \frac{54}{3}$$

$$= \frac{7}{3} \text{ units}^2$$



c) Find the area between $x = 0$ and $x = 4$

$$\text{Area} = \left[\frac{x^3}{3} - 2x^2 \right]_0^4$$

$$= \frac{(4)^3}{3} - 2(4)^2 - \left(\frac{(0)^3}{3} - 2(0)^2 \right)$$

$$= \frac{64}{3} - 32 - (0)$$

$$= \frac{64}{3} - \frac{96}{3}$$

$$= -\frac{32}{3}$$

Area is negative since below the x -axis

So Area = $\frac{32}{3} \text{ units}^2$

Area Under a Curve – a Note of Caution!!:

- Areas above the x -axis will be Positive, and Areas below the x -axis will be Negative.
- If part of the required area is above and part below the x -axis then these areas must be calculated separately and then added, ignoring any negative signs!!
- So for the Graph above the Area between zero and 5 would be given by: $A = \frac{7}{3} + \frac{32}{3} = \frac{39}{3} = 13$

Examples:

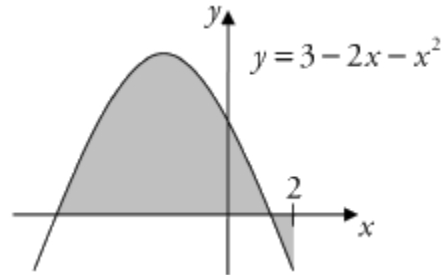
Ignore Negative!!

5. Find the shaded area shown here:

We need to find the Limits of Integration

In this case these will be the roots:

$$\begin{aligned} -x^2 - 2x + 3 &= 0 \\ -(x^2 + 2x - 3) &= 0 \\ (x - 1)(x + 3) &= 0 \\ x - 1 = 0 \text{ or } x + 3 = 0 \\ x &= 1 \text{ or } x = -3 \end{aligned}$$



$$\begin{aligned} \text{Area} &= \int_{-3}^1 -x^2 - 2x + 3 \, dx \\ &= \left[-\frac{x^3}{3} - \frac{2x^2}{2} + 3x \right]_{-3}^1 \\ &= \left[-\frac{x^3}{3} - x^2 + 3x \right]_{-3}^1 \\ &= -\frac{(1)^3}{3} - (1)^2 + 3(1) - \left(-\frac{(-3)^3}{3} - (-3)^2 + 3(-3) \right) \\ &= -\frac{1}{3} - 1 + 3 - \left(-\frac{-27}{3} - 9 - 9 \right) \\ &= -\frac{1}{3} + 2 - (9 - 18) \\ &= -\frac{1}{3} + 2 + 9 = \frac{32}{3} \text{ units}^2 \end{aligned}$$

$$\begin{aligned} \text{Area} &= \left[-\frac{x^3}{3} - x^2 + 3x \right]_1^2 \\ &= -\frac{(2)^3}{3} - (2)^2 + 3(2) - \left(-\frac{(1)^3}{3} - (1)^2 + 3(1) \right) \\ &= -\frac{8}{3} - 4 + 6 - \left(-\frac{1}{3} - 1 + 3 \right) \\ &= -\frac{8}{3} + 2 - \left(-\frac{1}{3} + 2 \right) \\ &= -\frac{7}{3} = \frac{7}{3} \text{ units}^2 \end{aligned}$$

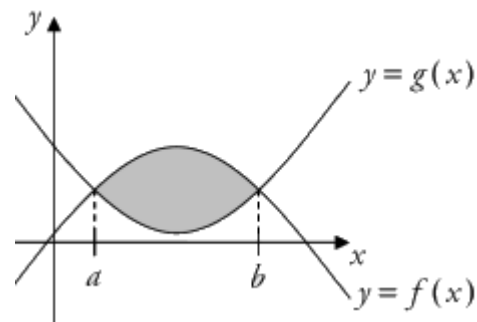
$$\text{Total Area} = \frac{7}{3} + \frac{32}{3} = \frac{39}{3} = 13 \text{ units}^2$$

Area Between 2 Curves:

- The Area between 2 Curves is given by:

$$\int_a^b f(x) - g(x) dx \text{ or } \int_a^b (\text{upper curve} - \text{lower curve}) dx$$

- The Limits, a and b , are normally the points of intersection of the 2 curves found by equating the 2 curves, i.e. $f(x) = g(x)$
- Areas above/below the x -axis do **NOT** need to be found separately here!
- If the curves swap positions (upper becomes lower & vice versa) then these areas need to be found separately and then added together.



Examples:

6. Find the shaded area shown here:

Limits of Integration:

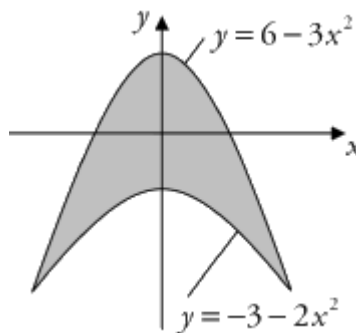
$$6 - 3x^2 = -3 - 2x^2$$

$$9 - x^2 = 0$$

$$(3 - x)(3 + x) = 0$$

$$3 - x = 0 \text{ or } 3 + x = 0$$

$$x = 3 \text{ or } x = -3$$



$$\text{Area} = \int_{-3}^3 9 - x^2 dx$$

Use the expression
equal to zero above

$$= \left[9x - \frac{x^3}{3} \right]_{-3}^3$$

$$= 9(3) - \frac{(3)^3}{3} - \left(9(-3) - \frac{(-3)^3}{3} \right)$$

$$= 27 - \frac{27}{3} - \left(-27 - \frac{-27}{3} \right)$$

$$= 27 - 9 - (-27 + 9)$$

$$= 18 - (-18)$$

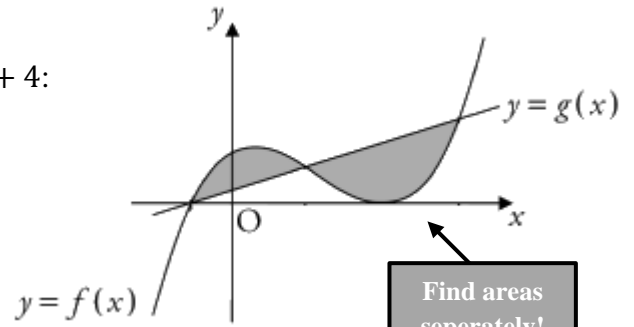
$$= 36 \text{ units}^2$$

7. Find the shaded area enclosed by
 $f(x) = x^3 - 7x^2 + 8x + 16$ and $g(x) = 4x + 4$:

Limits of Integration:

$$x^3 - 7x^2 + 8x + 16 = 4x + 4$$

$$x^3 - 7x^2 + 4x + 12 = 0$$



Synthetic Division to solve a cubic with factors of 12

$$\begin{array}{r|rrrr}
 2 & 1 & -7 & 4 & 12 \\
 & \downarrow & & & \\
 & 1 & -5 & -6 & 0
 \end{array}$$

Since Remainder = 0 $(x - 2)$ is a factor.

$$\text{Hence, } x^3 - 7x^2 + 4x + 12 = 0$$

$$\Rightarrow (x - 2)(x^2 - 5x - 6) = 0$$

$$\Rightarrow (x - 2)(x - 6)(x + 1) = 0$$

$$\Rightarrow x = -1, 2, 6$$

$$\text{Area} = \int_{-1}^2 x^3 - 7x^2 + 8x + 16 - (4x + 4) dx$$

$$= \int_{-1}^2 x^3 - 7x^2 + 4x + 12 dx$$

$$= \left[\frac{x^4}{4} - \frac{7x^3}{3} + 2x^2 + 12x \right]_{-1}^2$$

$$= \frac{(2)^4}{4} - \frac{7(2)^3}{3} + 2(2)^2 + 12(2) - \left(\frac{(-1)^4}{4} - \frac{7(-1)^3}{3} + 2(-1)^2 + 12(-1) \right)$$

$$= \frac{16}{4} - \frac{56}{3} + 8 + 24 - \left(\frac{1}{4} - \frac{-7}{3} + 2 - 12 \right)$$

$$= 36 - \frac{56}{3} - \left(\frac{1}{4} + \frac{7}{3} - 10 \right)$$

$$= 46 - \frac{63}{3} - \frac{1}{4}$$

$$= 46 - 21 - \frac{1}{4}$$

$$= 25 - \frac{1}{4}$$

$$= \frac{100}{4} - \frac{1}{4}$$

$$= \frac{99}{4}$$

$$= 24 \frac{3}{4} \text{ units}^2$$

$$\begin{aligned}
\text{Area} &= \int_2^6 4x + 4 - (x^3 - 7x^2 + 8x + 16) \, dx \\
&= \int_2^6 -x^3 + 7x^2 - 4x - 12 \, dx \\
&= \left[-\frac{x^4}{4} + \frac{7x^3}{3} - 2x^2 - 12x \right]_2^6 \\
&= -\frac{(6)^4}{4} + \frac{7(6)^3}{3} - 2(6)^2 - 12(6) - \left(-\frac{(2)^4}{4} + \frac{7(2)^3}{3} - 2(2)^2 - 12(2) \right) \\
&= \frac{-1296}{4} + \frac{1512}{3} - 72 - 72 - \left(-4 + \frac{56}{3} - 8 - 24 \right) \\
&= -324 + 504 - 144 - \left(-4 + \frac{56}{3} - 32 \right) \\
&= 72 - \frac{56}{3} \\
&= \frac{216}{3} - \frac{56}{3} \\
&= \frac{160}{3} = 53\frac{1}{3} \text{ units}^2
\end{aligned}$$

$$\begin{aligned}
\text{Total Area} &= \frac{160}{3} + \frac{99}{4} \\
&= \frac{640}{12} + \frac{297}{12} \\
&= \frac{937}{12} \\
&= 78\frac{1}{12} \text{ units}^2
\end{aligned}$$

8. A farmer is thinking of buying a new water trough for his cows.

His old trough holds 600 litres of water and he wants a new one which will hold at least this amount of water.

The trough is 3 metres long and the cross-section is made up of a Parabola, $y = x^2 - 7x + 11$ cut off by the line $y = 5$.

1 unit = 10cm.

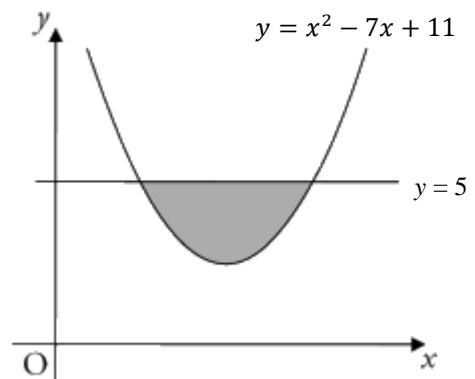
Should the farmer buy this new trough?



Limits of Integration:

$$\begin{aligned}
5 &= x^2 - 7x + 11 \\
-x^2 + 7x - 6 &= 0 \\
-(x^2 - 7x + 6) &= 0 \\
(x - 1)(x - 6) &= 0 \\
x - 1 = 0 \text{ or } x - 6 &= 0 \\
x = 1 \text{ or } x &= 6
\end{aligned}$$

Use this expression as the Integral



$$\begin{aligned}
\text{Area} &= \int_1^6 -x^2 + 7x - 6 \, dx \\
&= \left[-\frac{x^3}{3} + \frac{7}{2}x^2 - 6x \right]_1^6 \\
&= -\frac{(6)^3}{3} + \frac{7}{2}(6)^2 - 6(6) - \left(-\frac{(1)^3}{3} + \frac{7}{2}(1)^2 - 6(1) \right) \\
&= -\frac{216}{3} + 126 - 36 - \left(-\frac{1}{3} + \frac{7}{2} - 6 \right) \\
&= -72 + 90 - \left(-\frac{2}{6} + \frac{21}{6} - \frac{36}{6} \right) \\
&= 18 - \left(-\frac{17}{6} \right) \\
&= \frac{108}{6} - \left(-\frac{17}{6} \right) \\
&= \frac{125}{6} \\
&= 20\frac{5}{6} \text{ units}^2 && 1 \text{ unit} = 10\text{cm so } 1 \text{ unit}^2 = 100\text{cm}^2 \\
&= 20\frac{5}{6} \times 100 \text{ cm}^2 \\
&= 2083\frac{1}{3} \text{ cm}^2
\end{aligned}$$

$$\begin{aligned}
\text{So Volume} &= 2083\frac{1}{3} \times 300 \\
&= 625000 \text{ cm}^3 && 1\text{cm}^3 = 1\text{ml so } 1000\text{cm}^3 = 1 \text{ Litre} \\
&= 625 \text{ Litres}
\end{aligned}$$

So the farmer should buy this trough as $625 > 600$ Litres.

Differential Equations:

- Remember that Integration is the reverse of Differentiation.
- A Differential Equation is any equation which contains $\frac{dy}{dx}$ or $f'(x)$ e.g. $\frac{dy}{dx} = 4x - 5$
- In Higher we only consider first order differential equations
- To solve a Differential Equation we Integrate to find the General Solution which involves $+c$
- With more information, such as a point we can find the Particular Solution not involving c

Examples:

9. a) Find the Particular solution for the differential equation above if it passes through (2, 5):

$$\begin{aligned}\frac{dy}{dx} &= 3x - 5 && \Rightarrow y = \int 4x - 5 \, dx \\ & && \Rightarrow y = 2x^2 - 5x + c \quad \leftarrow \text{General Solution} \\ \text{When } x=2, y=5 & \Rightarrow 5 = 2(2)^2 - 5(2) + c \\ & \Rightarrow 5 = 8 - 10 + c \\ & \Rightarrow 5 = -2 + c \\ & \Rightarrow c = 7 \\ & \Rightarrow y = 2x^2 - 5x + 7 \quad \leftarrow \text{Particular Solution}\end{aligned}$$

- b) A function, f , defined on a suitable domain such that $f'(x) = x^3 - \frac{1}{x^2} + 3$.
Given that $f(2) = 7$ find $f(x)$ in terms of x

$$\begin{aligned}f'(x) &= x^3 - \frac{1}{x^2} + 3 && \Rightarrow f(x) = \int x^3 - \frac{1}{x^2} + 3 \, dx \\ & && \Rightarrow f(x) = \int x^3 - x^{-2} + 3 \, dx \\ & && \Rightarrow f(x) = \frac{x^4}{4} + x^{-1} + 3x + c \\ & && \Rightarrow f(x) = \frac{x^4}{4} + \frac{1}{x} + 3x + c \\ \text{When } x=2, f(2) &= 7 && \Rightarrow 7 = \frac{(2)^4}{4} + \frac{1}{2} + 3(2) + c \\ & && \Rightarrow 7 = 4 + \frac{1}{2} + 6 + c \\ & && \Rightarrow 7 = 10\frac{1}{2} + c \\ & && \Rightarrow c = -3\frac{1}{2} \\ & && \Rightarrow f(x) = \frac{x^4}{4} + \frac{1}{x} + 3x - 3\frac{1}{2}\end{aligned}$$