H Polynomials

SPTA Mathematics – Higher Notes

- Polynomials are expressions with two or more terms with different powers added together.
  e.g. $3x^5 + 7x^3 - 3x^2 + 6$
- The degree of the polynomial is given by the largest power, i.e. the polynomial above has degree 5.
- Constant terms like the 6 above can be written with a varaiable to the power zero, i.e. $6 = 6x^0$.
- Polynomials can be evaluated by substituting in a value for the variable, x in the example above.

Examples:

1. For the function above, $f(x) = 3x^5 + 7x^3 - 3x^2 + 6$, find $f(2)$

$$f(2) = 3(2)^5 + 7(2)^3 - 3(2)^2 + 6$$

$$f(2) = 3(32) + 7(8) - 3(4) + 6$$

$$f(2) = 96 + 56 - 12 + 6$$

$$f(2) = 146$$

Polynomial Division:

- We can divide Polynomials by other Polynomials, but in higher we only consider dividing by polynomials of degree1, i.e. linear expressions.

Examples:

2. $3x^5 + 7x^3 - 3x^2 + 6$ divided by $x - 2$

$$\begin{array}{r|l}
 & 3x^4 + 6x^3 + 19x^2 + 35x + 70 \\ \hline
x - 2 & 3x^5 + 0x^4 + 7x^3 - 3x^2 + 0x + 6 \\
 & 3x^5 - 6x^4 \\ \cline{2-2}
 & 6x^4 + 7x^3 - 3x^2 + 0x + 6 \\
 & 6x^4 - 12x^3 \\ \cline{2-2}
 & 19x^3 - 3x^2 + 0x + 6 \\
 & 19x^3 - 38x^2 \\ \cline{2-2}
 & 35x^2 + 0x + 6 \\
 & 35x^2 - 70x \\ \cline{2-2}
 & 70x + 6 \\
 & 70x - 140 \\ \cline{2-2}
 & 146
\end{array}$$

This is good to know but is not in the Higher course anymore!!

So $3x^5 + 7x^3 - 3x^2 + 6$ divided by $x - 2$

is $3x^4 + 6x^3 + 19x^2 + 35x + 70$ remainder 146

Synthetic Division:

- The above division can be done in a simpler way using Synthetic Division as follows:

$ax^4 + bx^3 + cx^2 + dx + e$ divided by $x - p$

Divisor

$$\begin{array}{c|ccccc} p & a & b & c & d & e \\ & \downarrow & ap & p(ap+b) & \dots & \dots \\ \hline & a & ap + b & \dots & \dots & \boxed{R} \end{array}$$

Quotient

Remainder

so *Polynomial* = (*Divisor*)(*Quotient*) + *Remainder*

Examples:

3. $3x^5 + 7x^3 - 3x^2 + 6$ divided by $x - 2$

$3x^5 + 0x^4 + 7x^3 - 3x^2 + 0x + 6$

$x - 2 = 0$
so $x = 2$

$$\begin{array}{c|cccccc} 2 & 3 & 0 & 7 & -3 & 0 & 6 \\ & \downarrow & 6 & 12 & 38 & 70 & 140 \\ \hline & 3 & 6 & 19 & 35 & 70 & \boxed{146} \end{array}$$

$3x^4 + 6x^3 + 19x^2 + 35x + 70$

Remainder is 146

So $3x^5 + 7x^3 - 3x^2 + 6$ divided by $x - 2$

is $3x^4 + 6x^3 + 19x^2 + 35x + 70$ remainder 146

Note: The answers to example 1, 2 & 3 above are all identical. This is because the 3methods are evaluating the same thing. So dividing by a linear expression is the same as evaluating a polynomial.

4. For the function, $f(x) = x^3 + x^2 - 22x - 40$, find $f(5)$:

By Substituting

$f(5) = (5)^3 + (5)^2 - 22(5) - 40$

$f(5) = 125 + 25 - 110 - 40$

$f(5) = 0$

OR

By Synthetic Division

$$\begin{array}{c|cccc} 5 & 1 & 1 & -22 & -40 \\ & \downarrow & 5 & 30 & 40 \\ \hline & 1 & 6 & 8 & \boxed{0} \end{array}$$

so $f(5) = 0$

Remainder/Factor Theorems:

- The REMAINDER THEOREM states that when a polynomial, $f(x)$, is divided by $(x - h)$ the remainder is $f(h)$.
- The FACTOR THEOREM states that if $f(h) = 0$ then $(x - h)$ is a Factor of the polynomial, $f(x)$.
- We can, therefore, use the Factor Theorem and Synthetic Division to factorise Polynomials. This also works for Quadratic Factorisation though it is best to do it the way shown in Nat 5!!
- Solving a Polynomial gives the roots of the Polynomial, i.e. where it crosses the x-axis.
- If the question asks you to factorise the polynomial do **NOT** go further and find the roots as this may lose you a mark!!
- We can also use the Factor Theorem and Synthetic Division to find unknown coefficients of terms in Polynomials.

Examples:

5. Show that $(x - 4)$ is a factor of $2x^4 - 9x^3 + 5x^2 - 3x - 4$

$$\begin{array}{r|rrrrr} 4 & 2 & -9 & 5 & -3 & -4 \\ & \downarrow & 8 & -4 & 4 & 4 \\ \hline & 2 & -1 & 1 & 1 & \boxed{0} \end{array}$$

Since Remainder = 0 $(x - 4)$ is a factor.

6. Show that $(x + 7)$ is a factor of $x^3 - 37x + 84$ and hence fully factorise the polynomial.

$$\begin{array}{r|rrrr} -7 & 1 & 0 & -37 & 84 \\ & \downarrow & -7 & 49 & -84 \\ \hline & 1 & -7 & 12 & \boxed{0} \end{array}$$

Since Remainder = 0 $(x + 7)$ is a factor.

$$\begin{aligned} \text{Hence, } x^3 - 37x + 84 &= (x + 7)(x^2 - 7x + 12) \\ &= (x + 7)(x - 3)(x - 4) \end{aligned}$$

7. Solve $2x^3 + 5x^2 - 28x - 15 = 0$

To solve a Polynomial firstly factorise it using factors of the numerical term → 15: ±1, ±3, ±5, ±15
You can try these numbers in any order!!

$$\begin{array}{r|rrrr} 1 & 2 & 5 & -28 & -15 \\ & \downarrow & 2 & 7 & -21 \\ \hline & 2 & 7 & -21 & \boxed{-36} \end{array}$$

Since Remainder ≠ 0
$(x - 1)$ is NOT a factor.

$$\begin{array}{r|rrrr} -1 & 2 & 5 & -28 & -15 \\ & \downarrow & -2 & -3 & 31 \\ \hline & 2 & 3 & -31 & \boxed{16} \end{array}$$

Since Remainder ≠ 0
$(x + 1)$ is NOT a factor.

$$\begin{array}{r|rrrr} 3 & 2 & 5 & -28 & -15 \\ & \downarrow & 6 & 33 & 15 \\ \hline & 2 & 11 & 5 & \boxed{0} \end{array}$$

Since Remainder = 0 $(x - 3)$ is a factor.

$$2x^3 + 5x^2 - 28x - 15 = 0$$
$$\Rightarrow \quad (x - 3)(2x^2 + 11x + 5) = 0$$
$$\Rightarrow \quad (x - 3)(2x + 1)(x + 5) = 0$$
$$\Rightarrow \quad x - 3 = 0, \ 2x + 1 = 0, \ x + 5 = 0$$
$$\Rightarrow \quad x = 3, \ x = -{}^1/_2, \ x = -5$$
$$\Rightarrow \quad x = -5, \ -{}^1/_2, \ 3$$

If the question asked to “Fully Factorise” the polynomial we would have stopped HERE and missed out the “= 0”

8. Find the roots of $2x^4 + 7x^3 - 10x^2 - 33x + 18$

$$\begin{array}{r|rrrrr} 1 & 2 & 7 & -10 & -33 & 18 \\ & \downarrow & 2 & 9 & -1 & -34 \\ \hline & 2 & 9 & -1 & -34 & \boxed{-16} \end{array}$$

Since Remainder $\neq 0$
$(x - 1)$ is <u>NOT</u> a factor.

$$\begin{array}{r|rrrrr} 2 & 2 & 7 & -10 & -33 & 18 \\ & \downarrow & 4 & 22 & 24 & -18 \\ \hline & 2 & 11 & 12 & -9 & \boxed{0} \end{array}$$

Since Remainder $= 0$
$(x - 2)$ is a factor.

So $2x^4 + 7x^3 - 10x^2 - 33x + 18 = (x - 2)(2x^3 + 11x^2 + 12x - 9)$

We will need to use Synthetic Division again to factorise this using → 9: ±1, ±3, ±9

$$\begin{array}{r|rrrr} 3 & 2 & 11 & 12 & -9 \\ & \downarrow & 6 & 102 & 684 \\ \hline & 2 & 17 & 114 & \boxed{675} \end{array}$$

Since Remainder $\neq 0$
$(x - 1)$ is <u>NOT</u> a factor.

$$\begin{array}{r|rrrr} -3 & 2 & 11 & 12 & -9 \\ & \downarrow & -6 & -15 & 9 \\ \hline & 2 & 5 & -3 & \boxed{0} \end{array}$$

Since Remainder $= 0$
$(x + 3)$ is a factor.

So $2x^4 + 7x^3 - 10x^2 - 33x + 18 \Longrightarrow (x - 2)(2x^3 + 11x^2 + 12x - 9) = 0$

$\Longrightarrow (x - 2)(x + 3)(2x^2 + 5x - 3) = 0$

$\Longrightarrow (x - 2)(x + 3)(2x - 1)(x + 3) = 0$

$\Longrightarrow (x - 2)(2x - 1)(x + 3)^2 = 0$

$\Longrightarrow x - 2 = 0,\ 2x - 1 = 0,\ x + 3 = 0$

$\Longrightarrow x = 2,\ x = {}^{1}/_{2},\ x = -3$

$\Longrightarrow x = -3,\ {}^{1}/_{2},\ 2$

If the question asked to Fully Factorise the polynomial we would have stopped HERE and missed out the "= 0"

9. If $(x - 3)$ is a factor of $x^3 - x^2 + px + 24$, find the value of p

$$\begin{array}{c|cccc} 3 & 1 & -1 & p & 24 \\ & \downarrow & 3 & 6 & 3p+18 \\ \cline{2-5} & 1 & 2 & p+6 & \boxed{3p+42} \end{array}$$

Notice the different sentence for this type of question!

Since $(x - 3)$ is a Factor the remainder is 0, so $3p + 42 = 0$

$$3p = -42$$

$$p = -14$$

10. When $f(x) = px^3 + 3x^2 - 17x + 4q$ is divided by $(x - 2)$ the remainder is 7, and if $(x - 1)$ is a factor of $f(x)$ then find the values of p and q.

$$\begin{array}{c|cccc} 1 & p & 3 & -17 & 4q \\ & \downarrow & p & p+3 & p-14 \\ \cline{2-5} & p & p+3 & p-14 & \boxed{p+4q-14} \end{array}$$

Since $(x - 1)$ is a Factor the remainder is 0, so $p + 4q - 14 = 0$

$$p + 4q = 14$$

$$\begin{array}{c|cccc} 2 & p & 3 & -17 & 4q \\ & \downarrow & 2p & 4p+6 & 8p-22 \\ \cline{2-5} & p & 2p+3 & 4p-11 & \boxed{8p+4q-22} \end{array}$$

Equals 7 NOT zero this time!!

so $8p + 4q - 22 = 7$

$$8p + 4q = 29$$

Use simultaneous equations from National 5 to fond the values of *p* and *q*

$5p + 4q = 14 \rightarrow$ ①

$8p + 4q = 29 \rightarrow$ ②

② – ① $3p = 15$

$p = 5$

Sub $p = 5$ into ①

$5(5) + 4q = 14$

$25 + 4q = 14$

$4q = -11$

$q = -2.75$

Curve Sketching:

- We can also use the Factor Theorem, Synthetic Division and Differentiation to sketch Polynomials:
  - Find the y-intercept when $x = 0$
  - Find the roots, when $y = 0$.
  - Find the Stationary Points and their Nature, when ${}^{dy}/_{dx} = 0$.
  - Sketch the curve.

Examples:

11. Sketch the curve $f(x) = x^3 + x^2 - 16x - 16$

Cuts y-axis when $x = 0$: $y = 0^3 + 0^2 - 16(0) - 16$

$y = -16$

Cuts x-axis when $y = 0$: $0 = x^3 + x^2 - 16x - 16$ ← **Factors of 16: ±1, ±2, ±4, ±8, ±16**

$$\begin{array}{r|rrr|r} -1 & 1 & 1 & -16 & -16 \\ & \downarrow & -1 & 0 & 16 \\ \hline & 1 & 0 & -16 & \boxed{0} \end{array}$$

Since Remainder = 0

$(x + 1)$ is a factor.

So $x^3 + x^2 - 16x - 16 = 0$

$\Rightarrow (x + 1)(x^2 - 16) = 0$

$\Rightarrow (x + 1)(x - 4)(x + 4) = 0$

$\Rightarrow x + 1 = 0,\ x - 4 = 0,\ x + 4 = 0$

$\Rightarrow x = -1,\ x = 4,\ x = -4$

$\Rightarrow x = -4,\ -1,\ 4$ ← **Roots**

Stationary points occur when ${}^{dy}/_{dx} = 0$: ← **Must be written**

$3x^2 + 2x - 16 = 0$

$\Rightarrow (3x + 8)(x - 2) = 0$

$\Rightarrow 3x + 8 = 0,\ x - 2 = 0$

$\Rightarrow x = -{}^{8}/_{3},\ x = 2$

x	→	$-{}^{8}/_{3}$	→	2	→
${}^{dy}/_{dx}$	+	0	–	0	+
	/	‾	\	_	/

Max @ $(-{}^{8}/_{3}, {}^{400}/_{27})$, Min @ $(2, -36)$

When $x = 2$:

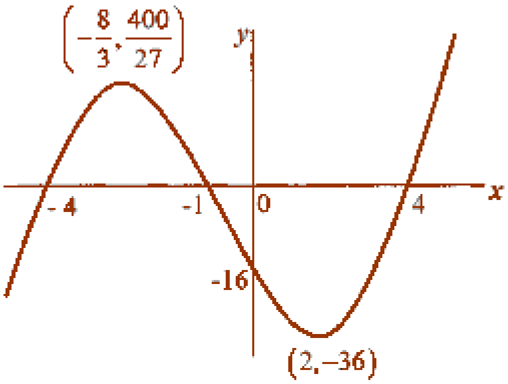
$y = 2^3 + 2^2 - 16(2) - 16$

$= -36$

When $x = -{}^{8}/_{3}$:

$y = (-{}^{8}/_{3})^3 + (-{}^{8}/_{3})^2 - 16(-{}^{8}/_{3}) - 16$

$= {}^{400}/_{27}$

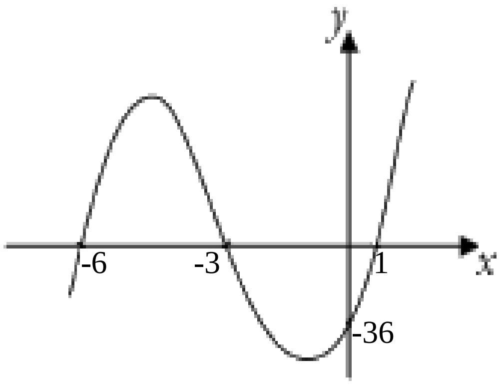


Functions from their Graphs:

- If we are given a Graph with the roots and one other point identified on it then we can find the functions equation as follows:
    - Turn the roots into factors: $x = a \implies (x - a)$
    - Express the function as: $f(x) = k(x - a)(x - b)(x - c)$, where a, b, c are roots and k is a constant.
    - Substitute in the other point to find the value of k.
    - Write the equation in full.
- If a stationary point lies on the x-axis then there are repeated roots, i.e. $(x - a)(x - a) = (x - a)^2$.

Examples:

12. Find the equation for the following graph:



Roots: $x = -6$, $x = -3$, $x = 1$

Factors: $(x + 6)$, $(x + 3)$, $(x - 1)$

Equation: $y = k(x + 6)(x + 3)(x - 1)$

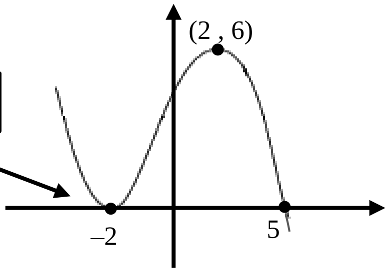
Point: $(0 , -36)$ so $x = 0$ and $y = -36$

Constant:
$$-36 = k(0 + 6)(0 + 3)(0 - 1)$$
$$-36 = k(6)(3)(-1)$$
$$-36 = -18k$$
$$k = 2$$

so equation is
$$y = 2(x + 6)(x + 3)(x - 1)$$
$$y = 2(x + 6)(x^2 + 2x - 3)$$
$$y = 2(x^3 + 8x^2 + 9x - 18)$$
$$y = 2x^3 + 16x^2 + 18x - 36$$

13. Find the equation for the following graph:

Repeated roots



Roots: $x = -2$, $x = 5$

Factors: $(x + 2)$, $(x - 5)$

Equation: $y = k(x + 2)(x + 2)(x - 5)$

Point: $(2 , 6)$ so $x = 2$ and $y = 6$

Constant:
$$6 = k(2 + 2)(2 + 2)(2 - 5)$$
$$6 = k(4)(4)(-3)$$
$$6 = -48k$$
$$k = -{}^{6}/_{48} = -{}^{1}/_{8}$$

so equation is
$$y = -{}^{1}/_{8}(x + 2)(x + 2)(x - 5)$$
$$y = -{}^{1}/_{8}(x + 2)(x^2 - 3x - 10)$$
$$y = -{}^{1}/_{8}(x^3 - x^2 - 16x - 20)$$
$$y = -{}^{1}/_{8}x^3 + {}^{1}/_{8}x^2 + 2x + {}^{5}/_{2}$$