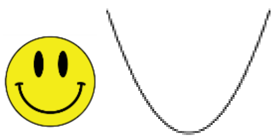
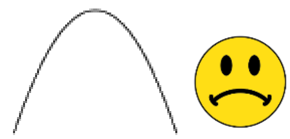
Quadratics

SPTA Mathematics – Higher Notes

- Quadratics are expressions with degree 2 and are of the form $ax^2 + bx + c$, where a $\neq$ 0.
- The Graph of a Quadratic is called a Parabola, and there are 2 types as shown below:



Happy Smiley Face
Minimum Turning Point
$a > 0$



Sad Smiley Face
Maximum Turning Point
$a < 0$

Examples:

1. For these functions state whether they have a Max or Min Turning Point:

a) $f(x) = 7x^2 - 3x + 6 \quad \Longrightarrow \quad a > 0$ so Minimum Turning Point

b) $f(x) = -x^2 + 5x - 8 \quad \Longrightarrow \quad a < 0$ so Maximum Turning Point

Sketching Parabolas:

- We can sketch a Parabola from its equation as follows:
  - State the type of Turning Point.
  - Find the y-intercept, $x = 0$.
  - Find the roots by factorising, $y = 0$.
  - Find the coordinates of the Turning Point, midway between the roots.

Examples:

2. Sketch the parabola of $y = x^2 + 2x - 8$

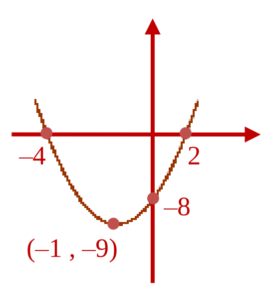
$a > 0$ so Minimum Turning Point

When $x = 0$: $\quad y = 0^2 + 2(0) - 8 = -8$

When $y = 0$: $\quad x^2 + 2x - 8 = 0$

$(x + 4)(x - 2) = 0$

$x = -4, 2$



Turning Point: $\quad x = \frac{-4+2}{2} = -1$, so $\quad y = (-1)^2 + 2(-1) - 8 = -9$

Solving Quadratic Inequalities:

- To solve a Quadratic Inequality you MUST do a QUICK sketch of the quadratic first but there is no need to find the coordinates of the Turning Point or the y-intercept

Examples:

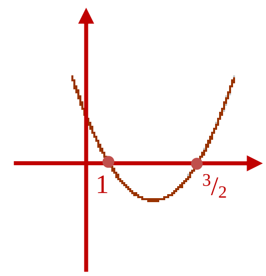
3. Solve $2x^2 - 5x + 3 > 0$

$a > 0$ so Minimum Turning Point

When $y = 0$: $2x^2 - 5x + 3 = 0$

$(2x - 3)(x - 1) = 0$

$x = 1, {}^3/_2$



So $2x^2 - 5x + 3 > 0$, when $x < 1$ and $x > {}^3/_2$

Note that $2x^2 - 5x + 3 < 0$ occurs when $1 < x < {}^3/_2$

4. Solve $6 + 7x - 3x^2 > 0$

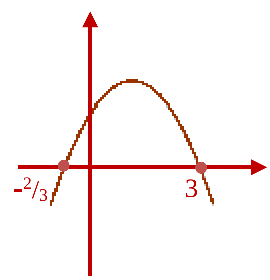
$a < 0$ so Maximum Turning Point

When $y = 0$: $-3x^2 + 7x + 6 = 0$

$-(3x^2 - 7x - 6) = 0$

$-(3x + 2)(x - 3) = 0$

$x = -{}^2/_3, 3$

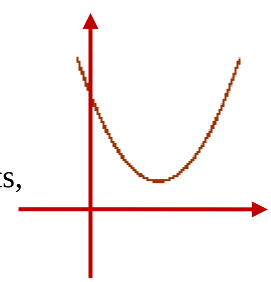


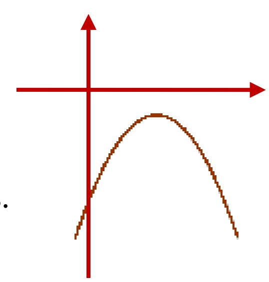
So $6 + 7x - 3x^2 > 0$, when $-{}^2/_3 < x < 3$

Note that $6 + 7x - 3x^2 < 0$ occurs when $x < -{}^2/_3$ and $x > 3$

Completing the Square:

- Completing the Square is a method for finding the Turning Point of a Quadratic which has no roots, i.e. it does not cross the x-axis. It can therefore **NOT** be factorised!!





- You saw how to Complete the Square in National 5 for simple quadratics.
- The coefficient of x^2 must be 1 before you start to Complete the Square.
- Completing the Square changes a quadratic expression $y = ax^2 + bx + c$ in to the form: $y = a(x + p)^2 + q$, if $a > 0$ and $y = q - a(x + p)^2$, if $a < 0$
- When in Completed Square form the Turning Point is $(-p, q)$

Examples:

5. Write $y = x^2 - 4x + 5$ in the form $y = (x + p)^2 + q$

$$x^2 - 4x + 5 = (x-2)^2 - (2)^2 + 5$$
$$= (x-2)^2 - 4 + 5$$
$$y = (x-2)^2 + 1$$

Half the coefficient of x to find p and then subtract p^2 then continue on the constant term. Now simplify!!

Turning Point is $(2, 1)$

6. Write $y = x^2 + 3x - 4$ in the form $y = (x + p)^2 + q$

$$x^2 + 3x - 4 = (x + {}^3/_2)^2 - ({}^3/_2)^2 - 4$$
$$= (x + {}^3/_2)^2 - {}^9/_4 - 4$$
$$= (x + {}^3/_2)^2 - {}^9/_4 - {}^{16}/_4$$
$$y = (x + {}^3/_2)^2 - {}^{25}/_4$$

Turning Point is $(-{}^3/_2, -{}^{25}/_4)$

7. Write $y = 7 + 6x - x^2$ in the form $y = q - (x + p)^2$

$$7 + 6x - x^2 = 7 + 6x - x^2$$
$$= -(x^2 - 6x - 7)$$
$$= -\left[(x-3)^2 - (3)^2 - 7\right]$$
$$= -\left[(x-3)^2 - 9 - 7\right]$$
$$= -\left[(x-3)^2 - 16\right]$$
$$y = 16 - (x-3)^2$$

Rearrange and then take out a common factor of –1

Complete the square of the expression inside the brackets then multiply out and simplify.

Turning Point is $(3, 16)$

8. Write $y = 4x^2 - 12x + 7$ in the form $y = a(x + p)^2 + q$

$$4x^2 - 12x + 7 = 4\left[x^2 - 3x\right] + 7$$
$$= 4\left[(x - {}^3/_2)^2 - ({}^3/_2)^2\right] + 7$$
$$= 4\left[(x - {}^3/_2)^2 - {}^9/_4\right] + 7$$
$$= 4(x - {}^3/_2)^2 - 9 + 7$$
$$y = 4(x - {}^3/_2)^2 - 2$$

Take out a common factor of 4 from the first 2 terms only.

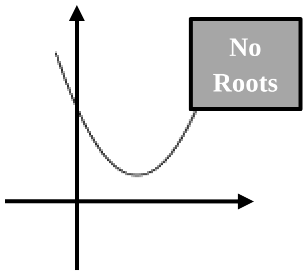
Complete the square of the expression inside the brackets then multiply out and simplify.

Turning Point is $({}^3/_2, -2)$

The Discriminant:

Must Know!!

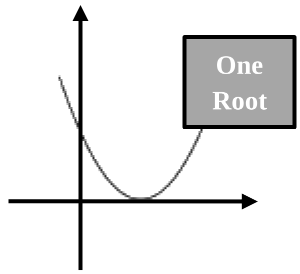
- We saw the Discriminant in National 5, $b^2 - 4ac$, part of the Quadratic Formula: $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$
- The discriminant is used to describe the Nature of a Quadratics Roots:



$b^2 - 4ac < 0$

No Real Roots exist

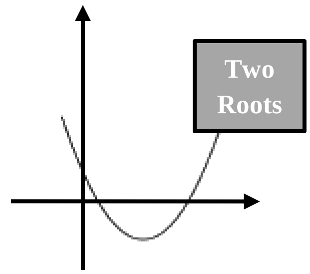
Cannot be Factorised



$b^2 - 4ac = 0$

Real & Equal Roots exist

Can be Factorised



$b^2 - 4ac > 0$

2 Real & Distinct Roots exist

May be Factorisable or may need to use the Quadratic Formula.

Note: For REAL roots, $b^2 - 4ac \geq 0$

Examples:

9. Find the Nature of the Roots of the function $f(x) = 9x^2 + 24x + 16$

$a = 9,\ b = 24,\ c = 16$

$b^2 - 4ac = 24^2 - 4(9)(16)$

$= 576 - 576$

$= 0$ since $b^2 - 4ac = 0$ there are REAL & EQUAL ROOTS.

10. Find the Nature of the Roots of the function $g(x) = -x^2 - 4x + 6$

$a = -1,\ b = -4,\ c = 6$

$b^2 - 4ac = (-4)^2 - 4(-1)(6)$

$= 16 + 24$

$= 40$ since $b^2 - 4ac > 0$ there are 2 REAL & DISTINCT ROOTS.

11. Find the Nature of the Roots of the function $h(x) = 12 + 8x + 10x^2$

$a = 10,\ b = 8,\ c = 12$

$b^2 - 4ac = 8^2 - 4(10)(12)$

$= 64 - 480$

$= -416$ since $b^2 - 4ac < 0$ there are No REAL ROOTS.

12. For what values of k does the equation $x^2 + kx + 9 = 0$ have equal roots?

$a = 1,\ b = k,\ c = 9$

Since Equal roots, $b^2 - 4ac = 0$ ← **Start with the sentence this time!!**

$k^2 - 4(1)(9) = 0$

$k^2 - 36 = 0$

$(k-6)(k+6) = 0$

So $k = 6\ , k = -6$

13. Find q given that the equation $6x^2 + 12x + q = 0$ has real roots.

$a = 6,\ b = 12,\ c = q$

Since Real roots, $b^2 - 4ac \geq 0$

$12^2 - 4(6)(q) \geq 0$

$144 - 24q \geq 0$

$-24q \geq -144$

$q \leq 6$ ← **Remember to switch the sign around if multiplying or dividing by a negative.**

14. Find m such that the equation $x^2 + (mx + 3)^2 - 3 = 0$ has equal roots, $m > 0$.

$$x^2 + (mx+3)^2 - 3 = x^2 + m^2x^2 + 6mx + 9 - 3$$
$$= x^2 + m^2x^2 + 6mx + 6$$
$$= (1 + m^2)x^2 + 6mx + 6$$

$a = 1 + m^2,\ b = 6m,\ c = 6$

Since Equal roots, $b^2 - 4ac = 0$

$(6m)^2 - 4(1+m^2)(6) = 0$

$36m^2 - 24 - 24m^2 = 0$

$12m^2 - 24 = 0$

$m^2 - 2 = 0$

$(m - \sqrt{2})(m + \sqrt{2}) = 0$

$m = \sqrt{2}\ ,$ ~~$m = -\sqrt{2}$~~ So $m = \sqrt{2}$

↑ **Question states $m > 0$ so this solution is not valid.**

15. Show that the equation $qx^2 + px - q = 0$, $q \neq 0$, has real roots.

$$a = q,\ b = p,\ c = -q$$

$$b^2 - 4ac = p^2 - 4(q)(-q)$$
$$= p^2 + 4pq^2$$

Since the sum of 2 squares can NEVER be negative, $p^2 + 4pq^2 \geq 0$
and therefore always has Real roots since $b^2 - 4ac \geq 0$

Functions from Parabolas:

- We can find the equation of of a Quadratic from its graph if we know 2 pieces of information
  - The Roots or the Turning Point.
  - One other point on the Graph.
- It is done in much the same ways as we saw in the Polynomials topic.

Examples:

16. Find the equation of the Parabola passing through the points: (1 , 0) , (6 , 0) , (0 , 3)

Roots: $x = 1,\ x = 6$

Factors: $(x - 1)$, $(x - 6)$

Equation: $y = k(x - 1)(x - 6)$

Point: (0 , 3) so $x = 0$ and $y = 3$

Constant: $3 = k(0 - 1)(0 - 6)$

$3 = k(-1)(-6)$

$3 = 6k$

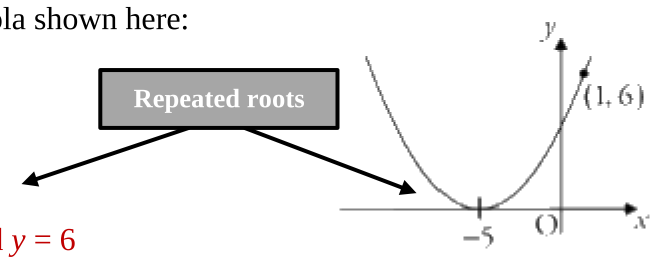
$k = \frac{1}{2}$

so equation is $y = \frac{1}{2}(x - 1)(x - 6)$

$y = \frac{1}{2}(x^2 - 7x + 6)$

$y = \frac{1}{2}x^2 - \frac{7}{2}x + 3$

17. Find the equation of the Parabola shown here:



Roots: $x = -5$

Factors: $(x + 5)$

Equation: $y = k(x + 5)(x + 5)$

Point: (1 , 6) so $x = 1$ and $y = 6$

Constant: $6 = k(1 + 5)(1 + 5)$

$6 = k(6)(6)$

$6 = 36k$

$k = \frac{1}{6}$

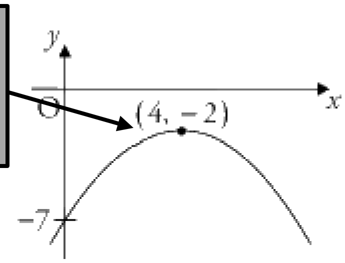
so equation is $y = \frac{1}{6}(x + 5)(x + 5)$

$y = \frac{1}{6}(x^2 + 10x + 25)$

$y = \frac{1}{6}x^2 + \frac{5}{3}x + \frac{25}{6}$

17. Find the equation of this Parabola:

No roots so use the Turning Point & Completed Square form of the equation.



Turning Point: (4 , –2)

Equation: $y = a(x-4)^2 - 2$

Point: (0 , –7) so $x = 0$ and $y = -7$

Constant: $-7 = a(0-4)^2 - 2$

$-5 = a(-4)^2$

$-5 = 16a$

$a = -{}^5/_{16}$

so equation is $y = -{}^5/_{16}(x-4)^2 - 2$

$y = -{}^5/_{16}(x^2 - 8x + 16) - 2$

$y = -{}^5/_{16}x^2 - {}^5/_{2}x - 5 - 2$

$y = -{}^5/_{16}x^2 - {}^5/_{2}x - 7$

Sketching Parabolas:

- We can sketch Parabolas as follows:
    - Find the y-intercept when $x = 0$
    - Use the Discriminant to determine the Nature of the roots.
    - Find the roots (if any), when $y = 0$ either by Factorising or Quadratic Formula.
    - Find the Turning Point, either Midway between roots or from Completed Square Form..
    - Determine the shape of the Parabola and sketch the curve.

Examples:

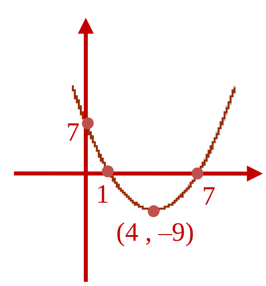
18. Sketch the graph $f(x) = x^2 - 8x + 7$

Cuts y-axis when $x = 0$:
$y = 0^2 - 8(0) + 7$
$y = 7$

Nature of Roots:
$a = 1,\ b = -8,\ c = 7$

$b^2 - 4ac = (-8)^2 - 4(1)(7)$
$= 64 - 28$
$= 36$ since $b^2 - 4ac > 0$ 2 REAL & DISTINCT ROOTS

Cuts x-axis when $y = 0$:
$0 = x^2 - 8x + 7$
$\Rightarrow (x-1)(x-7) = 0$
$\Rightarrow x - 1 = 0,\ x - 7 = 0$
$\Rightarrow x = 1,\ x = 7$



Turning Point (Midway):
$x = (1 + 7) \div 2 = 4$

$y = 4^2 - 8(4) + 7$
$y = -9$

Since $a > 0$, Minimum Turning Point

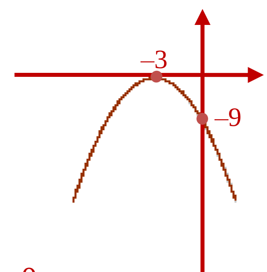
19. Sketch the graph $f(x) = -x^2 - 6x - 9$

Cuts y-axis when $x = 0$: $y = 0^2 - 6(0) - 9$
$y = -9$

Nature of Roots: $a = -1,\ b = -6,\ c = -9$

$b^2 - 4ac = (-6)^2 - 4(-1)(-9)$
$= 36 - 36$
$= 0$ since $b^2 - 4ac = 0$ REAL & EQUAL ROOTS

Cuts x-axis when $y = 0$: $0 = -x^2 - 6x - 9$
$\Rightarrow -(x^2 + 6x + 9) = 0$
$\Rightarrow -(x + 3)(x + 3) = 0$
$\Rightarrow x + 3 = 0$
$\Rightarrow x = -3$



Turning Point (Midway): Since a repeated root turning point lies on the x-axis when $x = -3$ and $y = 0$

Since $a < 0$, Maximum Turning Point

20. Sketch the graph $f(x) = 2x^2 - 8x + 13$

Cuts y-axis when $x = 0$: $y = 2(0)^2 - 8(0) + 13$
$y = 13$

Nature of Roots: $a = 2,\ b = -8,\ c = 13$

$b^2 - 4ac = (-8)^2 - 4(2)(13)$
$= 64 - 104$
$= -40$ since $b^2 - 4ac = 0$ NO REAL ROOTS

Turning Point (Comp □): $2x^2 - 8x + 13 = 2[x^2 - 4x] + 13$
$= 2[(x-2)^2 - (2)^2] + 13$
$= 2[(x-2)^2 - 4] + 13$
$= 2(x-2)^2 - 8 + 13$
$y = 2(x-2)^2 + 5$

Turning Point is $(2, 5)$

Since $a > 0$, Minimum Turning Point