H Rec. Relations

SPTA Mathematics – Higher Notes

- Recurrence Relations are a way of describing a sequence of numbers, like 3, 5, 7, 9, …
- A Recurrence Relation uses the previous term to describe(or calculate) the next term in the sequence.
- To properly describe a sequence using Recurrence Relations you should always state the Initial value.
- Linear Recurrence Relations are of the form:

$$U_{n+1} = aU_n + b \qquad U_0 = c$$

- U_{n+1}: **Next term in the sequence.**
- a: **Multiplier.**
- U_n: **Previous term in the sequence.**
- $U_0 = c$: **Initial value.**

Examples:

1. For the sequence described by $U_{n+1} = 3U_n - 10$, $U_0 = 15$, find U_4

$U_1 = 3U_0 - 10 = 3 \times 15 - 10 = 35$

$U_2 = 3U_1 - 10 = 3 \times 35 - 10 = 95$

$U_3 = 3U_2 - 10 = 3 \times 95 - 10 = 275$

$U_4 = 3U_3 - 10 = 3 \times 275 - 10 = \underline{\underline{815}}$

2. For the sequence above find the smallest value of n for which $U_n > 20000$

$U_5 = 3U_4 - 10 = 3 \times 815 - 10 = 2435$

$U_6 = 3U_5 - 10 = 3 \times 2435 - 10 = 7295$

$U_7 = 3U_6 - 10 = 3 \times 7295 - 10 = 21875$

So the smallest value of n for which $U_n > 20000$ is $\underline{\underline{7}}$

3. A patient is receiving treatment in Hospital.

She receives injections of a drug to help her condition every 4 hours.

She receives the first injection of 100ml at Midday on Monday.

Every 4 hours 30% of the drug leaves the bloodstream and a further 25ml is administered.

a) Construct a Recurrence Relation to describe the amount of the drug in her system at anytime:

30% of drug leaves bloodstream so 70% left: 70% → 0.7 ← **This does NOT need to be written**

$U_{n+1} = 0.7U_n + 25, \; U_0 = 100$

b) Use the Recurrence Relation to find the amount of the drug left in the blood after 24 hours:

4pm: $U_1 = 0.7U_0 + 25 = 0.7 \times 100 + 25 = 95$

8pm: $U_2 = 0.7U_1 + 25 = 0.7 \times 95 + 25 = 91.5$

Midnight: $U_3 = 0.7U_2 + 25 = 0.7 \times 91.5 + 25 = 89.05$

4am: $U_4 = 0.7U_3 + 25 = 0.7 \times 89.05 + 25 = 87.335$

8am: $U_5 = 0.7U_4 + 25 = 0.7 \times 87.335 + 25 = 86.1345$

Midday: $U_4 = 0.7U_3 + 25 = 0.7 \times 86.1345 + 25 = 85.29415$

So after 24 hours there will be 85.29ml of drugs in her system. ← **Always explain your answer.**

Limits:

- As n tends to infinity, usually written as $n \to \infty$, the terms in a sequence will do one of 3 things:
  - Continue to get larger and larger heading towards Infinity
  - Continue to get smaller and smaller heading towards negaitive infinity.
  - Converge towards a value, called a **LIMIT**.
- We are often asked to consider these convergent sequences.
- A Sequence given by $U_{n+1} = aU_n + b, \; U_0 = c$ will converge to a limit when: $-1 < a < 1$
- The limit is **NOT** dependant on the Initial Value.
- The Limit can be calculated in 2 ways , Algebraicly or using the formula: $L = \frac{b}{1-a}$
- Choose the method you prefer, the formula is not given on the Formulae Sheet in the exam.
- Questions will either ask: “Calculate the Limit” or “describe what happens in the long term” or something similar.

Examples:

4. Calculate the Limit for the sequence in Example 3 above.

A limit exists since $-1 < 0.7 < 1$, Let L be the Limit.

Algebraicly		**Formula**
$L = 0.7L + 25$		$L = \frac{b}{1-a}$
$0.3L = 25$	**OR**	$L = \frac{25}{1-0.7}$
$L = \frac{25}{0.3} = 83.33\text{ml}$		$L = \frac{25}{0.3} = 83.33\text{ml}$

So in the long term if the current course of treatment continues the amount of the drug will settle at but not go below 83.33ml.

For Limit Questions a statement must always be written.

5. The population of Deer in Sherwood Forest at the start of 2016 is 300.
The population is expected to drop by 8% each year.
To combat this fall it is decided to introduce a further 20 deer on the 1st of January each year.

a) How many deer will be in the Forest on the 31st December 2019 years.

8% drop in deer population so 92% left: 92% → 0.92

$U_{n+1} = 0.92U_n + 20$, $U_0 = 300$

1st Jan 2017: $U_1 = 0.92U_0 + 20 = 0.92 \times 300 + 20 = 296$

1st Jan 2018: $U_2 = 0.92U_1 + 20 = 0.92 \times 296 + 20 = 292.32$

1st Jan 2019: $U_2 = 0.92U_1 + 20 = 0.92 \times 292.32 + 20 = 288.9344$

31st Dec 2019: $U_3 = 0.92U_2 = 0.92 \times 288.9344 = 265.82 \approx$ <u>265 deer</u>

Don't add the 20 on here.

b) Over the long term will the population of Deer ever fall below 240 deer?

A limit exists since $-1 < 0.92 < 1$, Let L be the Limit.

$L = 0.92L + 20$		$L = \frac{b}{1-a}$
$0.08L = 20$	**OR**	$L = \frac{20}{1-0.92}$
$L = \frac{20}{0.08} = 250\text{ deer}$		$L = \frac{20}{0.08} = 250\text{ deer}$

Be very careful when asked to explain your answer.

In the long term the number of deer in the forest will increase towards but not exceed 250 deer. However the population of deer **will** fail below240 deer to 230 deer just before the 20 new deer are introduced on the 1st of January each year.

6. A sequence is defined by $U_{n+1} = aU_n + b$, with $U_1 = 4$, $U_2 = 3.6$ and $U_3 = 2.04$.
Find a and b and then find U_6

$U_2 = aU_1 + b \Longrightarrow 36 = a \times 4 + b \quad \Longrightarrow \quad 4a + b = 3.6 \rightarrow$ ①

$U_3 = aU_2 + b \Longrightarrow 20.4 = a \times 3.6 + b \quad \Longrightarrow \quad 3.6a + b = 2.04 \rightarrow$ ②

① – ② $\Longrightarrow \quad 0.4a = 1.56$

$a = 3.9$

Sub $a = 3.9$ into ① $\Longrightarrow \quad 4 \times 3.9 + b = 3.6$

$15.6 + b = 3.6$

$b = -12$

So $U_{n+1} = 3.9U_n - 12 \Longrightarrow U_4 = 3.9U_3 - 12 = 3.9 \times 2.04 - 12 = -4.044$

$U_5 = 3.9U_4 - 12 = 3.9 \times (-4.044) - 12 = -27.772$

$U_6 = 3.9U_5 - 12 = 3.9 \times (-27.772) - 12 = \underline{\underline{-120.31}}$