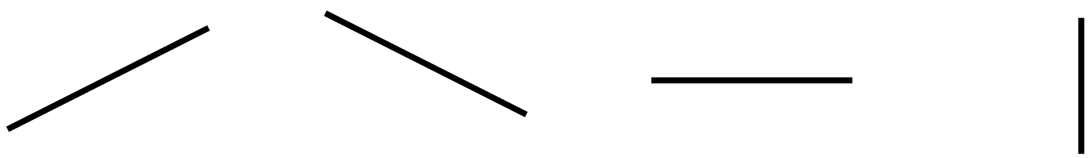
Straight Line

SPTA Mathematics – Higher Notes

Gradient – From National 5:

- Gradient is a measure of a lines slope, the greater the gradient the more steep its slope and vice versa.
- We use the letter *m* to represent Gradient.
- Gradient of a Straight Line joining 2 points $A\,(x_A\,,\,y_A)$ and $B\,(x_B\,,\,y_B)$ is given by: $m_{AB}=\frac{y_B-y_A}{x_B-x_A}$
- Two lines with the **SAME** Gradient are said to be **PARALLEL**.
- Gradient, *m*, can be found from a line's equation by rearranging it into the form $y = mx + c$
- The Gradient of a line lets us know the direction of the line as follows:



Slopes UP from Left to Right	Slopes DOWN from Left to Right	HORIZONTAL	VERTICAL
Positive Gradient	Negative Gradient	Zero Gradient	Undefined Gradient
$y = mx + c$	$y = mx + c$	$y = b$	$x = a$

Examples:

1. Find the Gradient of the Straight Line joining the points $\underset{x_A\ \ y_A}{A(0\,,\,-1)}$ and $\underset{x_B\ \ y_B}{B(2\,,\,3)}$

It is a good idea to label your points.

$$m_{AB}=\frac{y_B-y_A}{x_B-x_A}$$
$$=\frac{3-(-1)}{2-0}$$
$$=\frac{4}{2}=2$$

2. The line through the points $\underset{x_P\ \ y_P}{P(-2\,,\,5)}$ and $\underset{x_Q\ \ y_Q}{Q(7\,,\,a)}$ has gradient 3. What is the value of *a*?

$$m_{PQ}=\frac{y_Q-y_P}{x_Q-x_P}=\frac{a-5}{7-(-2)}$$

$$\Rightarrow \frac{a-5}{9}=3$$
$$a-5=27$$
$$a=32$$

3. Find the Gradient of the line parallel to $2y + 3x - 5 = 0$

$2y + 3x - 5 = 0$

$2y = -3x + 5$

$y = -{}^{3}/_{2}\, x + {}^{5}/_{2}$

so $m = -{}^{3}/_{2}$ hence Parallel Gradient is $-{}^{3}/_{2}$

Exercise 1:

In your jotters answer the following questions:

1. Use the gradient formula to determine the gradients of the lines joining the following points:

a) $A(2, 1)$, $B(8, 3)$ **(b)** $C(0, 3)$, $D(10, 8)$ **(c)** $E(1, -1)$, $F(3, 5)$

d) $G(-1, -3)$, $H(2, 4)$ **(e)** $I(4, 1)$, $J(-1, 3)$ **(f)** $K(2, -3)$, $L(-2, 5)$

g) $M(-1, -2)$, $N(-1, 6)$ **(h)** $P(5, 3)$, $Q(-1, 3)$

2. What types of Line are **(g)** and **(h)** in question **1** above?

3. State the direction of slope for each of the lines in Question **1 (a)** to **(f)** above

4. Find the gradients of the two lines joining the points $A(-5, -2)$ to $B(3, 2)$ and $W(1, -5)$ to $Z(5, -3)$

What do your two answers tell you about the line AB and the line XZ?

5. Determine the values of p, q and r, when:

a) $A(3, 1)$, $B(5, p)$, and $m_{AB} = 4$

b) $L(4, -1)$, $M(0, q)$, and LM is parallel to the line joining $C(1, 3)$ to $D(-1, 7)$.

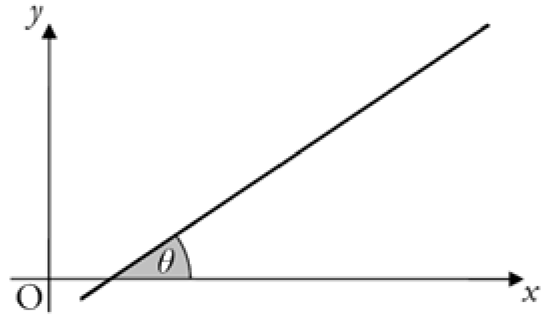
c) $S(-1, -2)$, $T(5, r)$, and ST is parallel to the line with equation $-4x + 8y + 5 = 0$

6. It is known that $IJKL$ is a parallelogram, where $I(1, 3)$, $J(-1, 1)$ and $K(-2, 5)$.

Find the gradients of the two lines IL and KL. (A sketch may help).

Gradient – New Stuff:

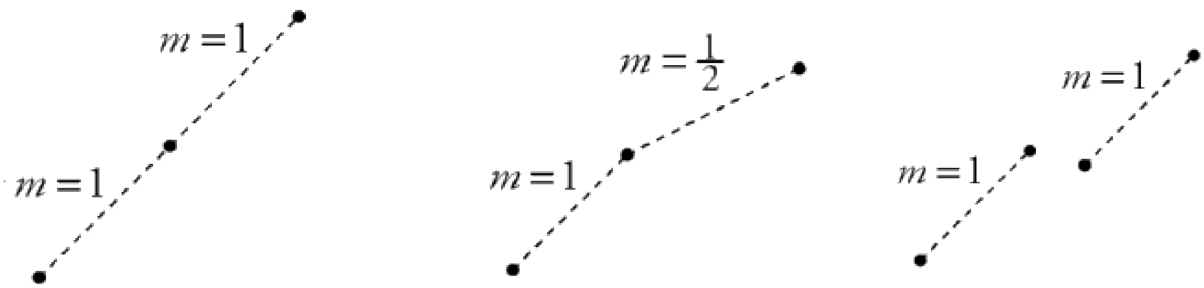
- The Gradient of a line can also be found using m = Tan θ, where θ is the angle the line makes with the **positive direction of the *x*-axis** as follows:



- If 2 lines, with Gradients m_1 and m_2, are perpendicular (meet at 90°) then: $m \times m_{perp} = -1$

  So if you know the gradient of one line you can find the Gradient of the perpendicular line by inverting (flipping) it and changing the sign.

- For horizontal lines, i.e. $m = 0$ then the perpendicular line will be vertical with Undefined Gradient and vice versa.
- Points which lie on the same straight line are said to be collinear as follows:



Equal Gradients with a common point COLLINEAR	Unequal Gradients with a common point NOT Collinear	Equal Gradients but NO common point NOT Collinear

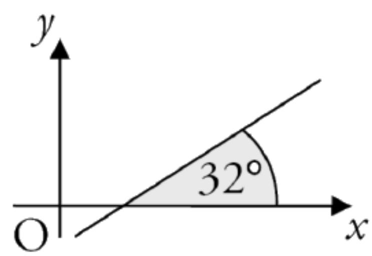
Examples:

4. Calculate the Gradient of the straight line opposite:

m = Tan θ

= Tan 32°

= 0.62 to 2 d.p.



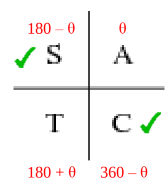
5. Find the angle that the line through the points $P(\overset{x_P}{4}, \overset{y_P}{-3})$ and $Q(\overset{x_Q}{1}, \overset{y_Q}{6})$ makes with the positive direction of the x-axis.

$m_{PQ} = \frac{y_Q - y_P}{x_Q - x_P}$ $\Rightarrow$ m = Tan θ

$= \frac{6-(-3)}{1-4}$ -3 = Tan θ

$= \frac{9}{-3} = -3$ $\theta = \text{Tan}^{-1}(-3)$

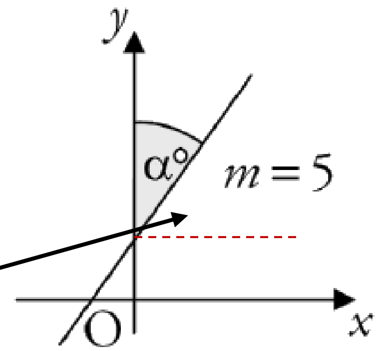
$\theta = 180 - 71.57° = 108.43°$ to 2 d.p.



Remember if the Gradient is a negative, do not enter this into your calculator. Use the positive gradient along with the CAST diagram from National 5 to find the correct angle.

The Angle will be less than 180°

6. Find the angle, α, marked in the diagram opposite:



$m = \text{Tan}\,\theta$

$5 = \text{Tan}\,\theta$

$\theta = \text{Tan}^{-1}\,5$

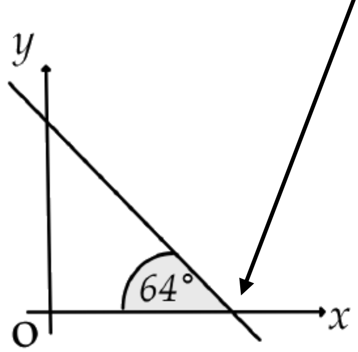
$\theta = 78.69°$ to 2 d.p.

so $\alpha = 90 - 78.69$

$\alpha = 11.31°$ to 2 d.p.

m = Tan θ uses this angle.

7.



Find the Gradient of the line shown.

$\theta = 180 - 64$

$\theta = 116°$

so $m = \text{Tan}\,\theta$

$m = \text{Tan}\,116$

$m = -2.05$ to 2 d.p.

8. State the perpendicular gradient of:

a) $m = -3$, since perpendicular

$m \times m_{perp} = -1$

so $m_{perp} = {}^{1}/_{3}$

(b) $m = {}^{2}/_{5}$, since perpendicular

$m \times m_{perp} = -1$

so $m_{perp} = -{}^{5}/_{2}$

Statement MUST be written.

9. Find the Gradient of the line perpendicular to the line joining the points $S(1, -2)$ and $T(-4, 5)$.

$x_S \;\; y_S \qquad x_T \;\; y_T$

$$m_{ST} = \frac{y_T - y_S}{x_T - x_S}$$

$$= \frac{5-(-2)}{-4-1}$$

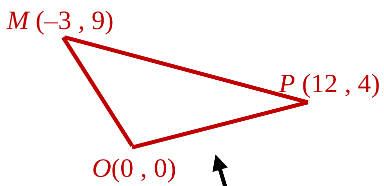
$$= -\frac{7}{5}$$

Since perpendicular

$m_{ST} \times m_{perp} = -1$

so $m_{perp} = {}^{5}/_{7}$

10. The points $M\,(-3\,,9)$, $P(12\,,4)$ and the Origin form a triangle, show that it's a right angled triangle.
$\quad x_M \; y_M \qquad x_P \; y_P$



A quick sketch may make the question clearer.

$$m_{MP}=\frac{y_P-y_M}{x_P-x_M}=\frac{4-9}{12-(-3)}=\frac{-5}{15}=-\frac{1}{3}$$

$$m_{MO}=\frac{y_O-y_M}{x_O-x_M}=\frac{0-9}{0-(-3)}=\frac{-9}{3}=-3$$

$$m_{PO}=\frac{y_O-y_P}{x_O-x_P}=\frac{0-4}{0-12}=\frac{-5}{-12}=\frac{1}{3}$$

Since m_{MO} x $m_{PO}=-1$, MO is perpendicular to PO
Therefore Triangle MOP is right angled at O.

11. Show that the points $P\,(-6\,,-1)$, $Q(0\,,2)$ and $R\,(8\,,6)$ are collinear.
$\quad x_P \; y_P \qquad x_Q \; y_Q \qquad x_R \; y_R$

$$m_{PQ}=\frac{y_Q-y_P}{x_Q-x_P}=\frac{2-(-1)}{0-(-6)}=\frac{3}{6}=\frac{1}{2}$$

$$m_{QR}=\frac{y_R-y_Q}{x_R-x_Q}=\frac{6-2}{8-0}=\frac{4}{8}=\frac{1}{2}$$

Statements MUST be written to gain full marks.

Since $m_{PQ}=m_{QR}$, PQ is parallel to QR and
Since Q is a common point, P, Q, R are collinear.

MIA Pg 4 Ex 2A Qu 1, 3, 7 & 8 Pg 5 Ex 2B Qu 1, 2, 4 & 5 Pg 7 Ex 3 Qu 4 – 8(a) & 9

Straight Line Formulae:

- As well as the 2 Gradient formulae mentioned above you will need to remember the following Formulae as they will **NOT** be given on your Formulae Sheet in the exam.
- From National 5 you should already know the Straight Line Formula: $y-b=m(x-a)$
- Remember to be able to find the equation of a Straight Line you need to know 2 things: A Point on the line and the Gradient of the line or 2 points on the line.
- The Straight Line Formulae will not work if the Gradient is undefined (Denominator = 0), i.e. a Vertical Line which has equation $x=a$
- The distance between 2 points $A\,(x_A\,,y_A)$ and $B\,(x_B\,,y_B)$ is given by the Distance Formula: $AB=\sqrt{(x_B-x_A)^2+(y_B-y_A)^2}$
- The Midpoint between 2 points $A\,(x_A\,,y_A)$ and $B\,(x_B\,,y_B)$ is given by the Midpoint Formula: $\text{Mid}_{AB}=\left(\frac{x_A+x_B}{2}\,,\,\frac{y_A+y_B}{2}\right)$

Examples:

12. a) Find the equation of the straight line joining the points $A(-1, -3)$ and $B(2, 3)$

$x_A \quad y_A \qquad x_B \quad y_B$

$$m_{AB} = \frac{y_B - y_A}{x_B - x_A} = \frac{3-(-3)}{2-(-1)} = \frac{6}{3} = 2$$

$y - b = m(x - a)$

$y - 3 = 2(x - 2)$

You can sub in either point above to get the same equation.

$y - 3 = 2x - 4$

The final equation can be left in any format.

$y = 2x - 1$ or $y - 2x = -1$ or $y - 2x + 1 = 0$

$x \quad y$

b) Does the point $T(-4, -9)$ lie on the line AB above?

$y = 2x - 1$

$y = 2(-4) - 1$

$y = -8 - 1$

$y = -9$ hence the point T lies on the line $y = 2x - 1$

13. Find the equation of the straight line joining the points $A(2, -1)$ and $B(2, 5)$

$x_A \quad y_A \qquad x_B \quad y_B$

$$m_{AB} = \frac{y_B - y_A}{x_B - x_A} = \frac{5-(-1)}{2-2} = \frac{6}{0} = Undefined$$

Vertical line.

$x = a$

$x = 2$

14. Find the Equation of the line perpendicular to the line joining the points $S(1, -2)$ and $T(-4, 5)$ and passing through the point $U(-3, 2)$.

$x_S \quad y_S \qquad x_T \quad y_T$

$$m_{ST} = \frac{y_T - y_S}{x_T - x_S} = \frac{5-(-2)}{-4-1} = -\frac{7}{5}$$

Since perpendicular

$m_{ST} \times m_{perp} = -1$

so $m_{perp} = {}^5/_7$

$y - b = m(x - a)$

$y - 2 = {}^5/_7(x - (-3))$

$7y - 14 = 5x + 15$

$7y = 5x + 29$

15. Calculate the length of the line joining the points $A(1, -2)$ and $B(-3, 6)$.

$x_A \quad y_A \qquad x_B \quad y_B$

$$AB = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$$

$$AB = \sqrt{(-3-1)^2 + (6-(-2))^2}$$

$$AB = \sqrt{(-4)^2 + 8^2}$$

$$AB = \sqrt{16 + 64}$$

$$AB = \sqrt{80}$$

$$AB = \sqrt{16 \times 5} = 4\sqrt{5} \text{ units}$$

Preferable to leave as a SURD rather than a decimal

16. Find the midpoint of the 2 coordinates in Example 15 above.

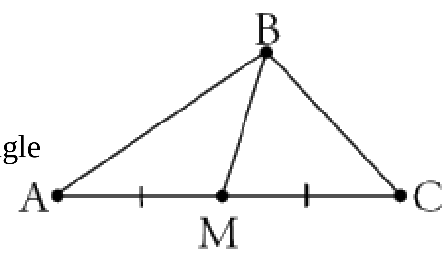
$$\text{Mid}_{AB} = \left(\frac{x_A + x_B}{2} \ , \ \frac{y_A + y_B}{2}\right)$$

$$\text{Mid}_{AB} = \left(\frac{1+(-3)}{2} \ , \ \frac{-2+6}{2}\right)$$

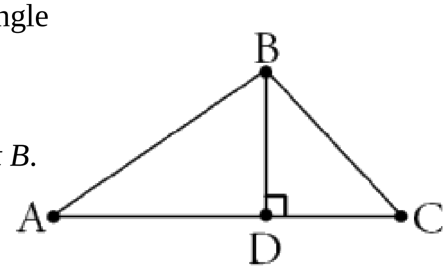
$$\text{Mid}_{AB} = \left(\frac{-2}{2} \ , \ \frac{4}{2}\right) = (-1\,, 2)$$

MIA Pg 2 Ex 1 Qu 3, 5, 7, 10 & 4 Pg 8 Ex 4 Qu 3 – 10 Pg 11 Ex 6 Qu 2 – 6

3 Special Lines:

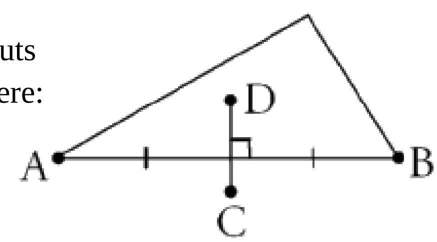


- A **MEDIAN** is a straight Line through a Vertex of a Triangle to the **Midpoint** of the opposite side as shown here:
- To find the equation of the MEDIAN, first find the Midpoint, then the Gradient of *BM* and then find the equation using either *Pt B* or *Pt M*.

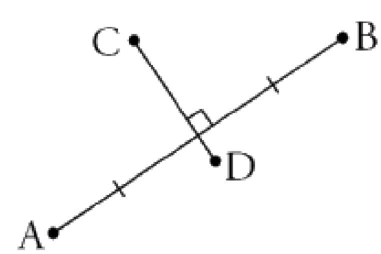


- An **ALTITUDE** is a straight Line through a Vertex of a Triangle **Perpendicular** to the opposite side as shown here:
- To find the equation of the Altitude, first find the the Gradient of *AC*, flip it and then find the equation using *Pt B*.

  There is NO need to find *Pt D* in order to find the Altitude!



- A **PERPENDICULAR BISECTOR** is a straight Line which cuts through the Midpoint of another line at right angles as shown here:



- Perpendicular Bisectors do **NOT** need to involve Triangles.
- To find the equation of the Perpendicular Bisectors, first find the the Gradient of *AB*, flip it, now find the Midpoint of *AB* and then find the equation using the midpoint.

- When two lines cross over they are said to Intersect. We can find the Point of Intersection, POI, by using Simultaneous Equations which you saw in National 5.
- There are 3 ways of finding the POI, Elimination, Substitution and by Equating.
- If a line crosses the other line in the middle it is said to **BISECT** the line.

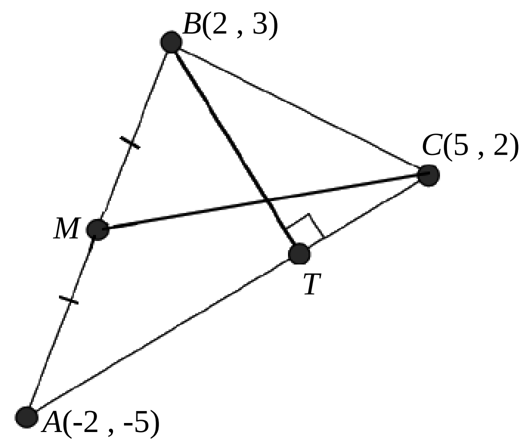
Examples:

17. **a)** Find the equation of the MEDIAN from *Pt C*.

b) Find the equation of the ALTITUDE from *Pt B*.

c) Find the point where these 2 lines intersect.

d) Find the PERPENDICULAR BISECTOR of *BC*.



a) $\text{Mid}_{AB} = \left(\frac{x_A+x_B}{2} , \frac{y_A+y_B}{2}\right)$ $\quad m_{MC} = \frac{y_C-y_M}{x_C-x_M}$ $\quad y - b = m(x - a)$

$\text{Mid}_{AB} = \left(\frac{-2+2}{2} , \frac{-5+3}{2}\right)$ $\quad = \frac{2-(-1)}{5-0}$ $\quad y - 2 = {}^3/_5(x - 5)$

$\text{Mid}_{AB} = \left(\frac{0}{2} , \frac{-2}{2}\right) = (0 , -1)$ $\quad = \frac{3}{5}$ $\quad 5y - 10 = 3x - 15$

$5y - 3x = -5$

b) $m_{AC} = \frac{y_C-y_A}{x_C-x_A}$ $\quad$ Since perpendicular $\quad y - b = m(x - a)$

$= \frac{2-(-5)}{5-(-2)}$ $\quad m_{AC} \text{ x } m_{perp} = -1$ $\quad y - 2 = -1(x - 3)$

$= \frac{7}{7} = 1$ $\quad$ so $m_{perp} = -1$ $\quad y - 2 = -x + 3$

$y + x = 5$

c) $5y - 3x = -5 \rightarrow 1$ $\quad \Rightarrow \quad 5y - 3x = -5 \rightarrow 1$ $\quad$ Sub $y = 1.25$ into 2

$y + x = 5 \rightarrow 2 \text{ x } 3$ $\quad \Rightarrow \quad 3y + 3x = 15 \rightarrow 3$ $\quad 1.25 + x = 5$

$1 + 3 \quad 8y = 10$ $\quad x = 3.75$

$y = 1.25$ $\quad$ POI (3.75 , 1.25)

d) $\text{Mid}_{BC} = \left(\frac{x_B+x_{BC}}{2} , \frac{y_B+y_C}{2}\right)$ $\quad m_{BC} = \frac{y_C-y_B}{x_C-x_B}$ $\quad y - b = m(x - a)$

$\text{Mid}_{BC} = \left(\frac{2+5}{2} , \frac{3+2}{2}\right)$ $\quad = \frac{2-3}{5-2}$ $\quad y - {}^5/_2 = 3(x - 4)$

$\text{Mid}_{BC} = \left(\frac{8}{2} , \frac{5}{2}\right) = (4 , \frac{5}{2})$ $\quad = -\frac{1}{3}$ $\quad y - {}^5/_2 = 3x - 12$

Since Perpendicular $\quad 2y - 5 = 6x - 24$

$m_{BC} \text{ x } m_{perp} = -1$ $\quad 2y - 6x = -19$

so $m_{perp} = 3$

<u>Note:</u>

Part **(c)** above was completed using the ELIMINATION method which is probably the one you are most comfortable with using. It could also have been solved using SUBSTITUTION as follows:

$5y - 3x = -5 \rightarrow (1)$	Sub $y = -x + 5$ into (1)	Sub $x = 3.75$ into (2)
$y + x = 5$	$\Rightarrow\ 5(-x + 5) - 3x = -5$	$y + 3.75 = 5$
$y = -x + 5 \rightarrow (2)$	$\Rightarrow\ -5x + 25 - 3x = -5$	$y = 1.25$
	$\Rightarrow\ -8x = -30$	
	$\Rightarrow\ x = 3.75$	POI $(3.75 , 1.25)$

The third method of EQUATING the 2 equations would not be suitable for this question.
An example of the Equating Method is shown below.
This method is only suitable when both equations can be expressed as $y =$ or $x =$.

18. $y = 3x + 6$

$y = -2x - 4$

$3x + 6 = -2x - 4$
$5x = -10$
$x = -2$

Sub $x = -2$ into either starting equation
$y = 3(-2) + 6$
$y = 0$
POI $(-2 , 0)$

MIA Pg 11 Ex 6 Qu 7 – 13 Pg 14 Ex 7A Qu All Pg 15 Ex 7B Qu All