Vectors

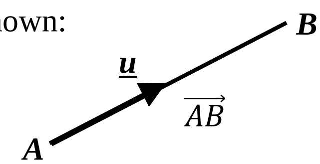
SPTA Mathematics – Higher Notes

Definitions:

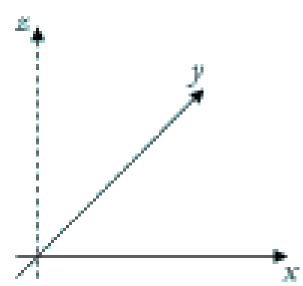
- MAGNITUDE - the SIZE of an object
- SCALAR - quantities or objects that have magnitude only,
  i.e. Length, Weight, Number of Sharks in the sea, Height of a wall, etc.
- VECTOR - quantities or objects that have magnitude <u>AND</u> direction,
  i.e. A plane flys 300km on a bearing of 145°
  Wind speed is 15mph Southeasterly.

Vectors:

- A vector can be named in 2 ways meaning the same thing:
  - A Bold, Italisised and underlined letter, $\underline{\boldsymbol{u}}$ represents Vector $\underline{\boldsymbol{u}}$
  - 2 capital letters with an arrow above, $\overrightarrow{AB}$, represents the Directed Line Segment AB.
- A vector can also be shown graphically as shown:



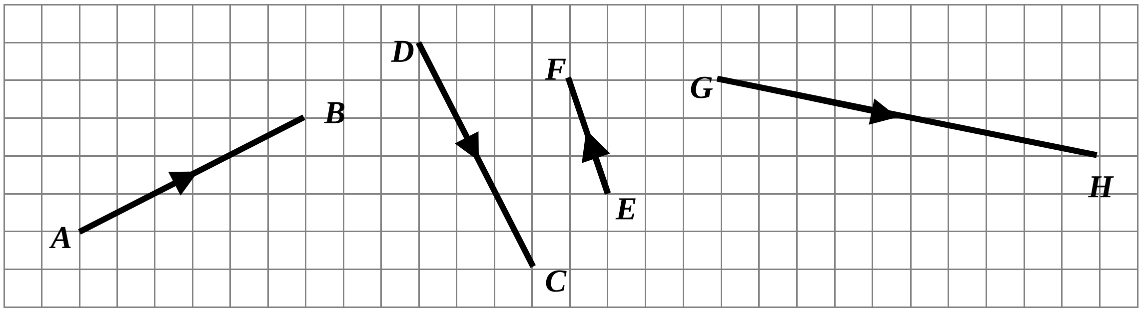
- A vector can be described using components, in:
  - 2D $\underline{\boldsymbol{u}} = \begin{pmatrix} x \\ y \end{pmatrix}$, e.g. $\underline{\boldsymbol{u}} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$ meaning 3 RIGHT and 4 DOWN.
  - 3D $\underline{\boldsymbol{u}} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, e.g. $\underline{\boldsymbol{u}} = \begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix}$ meaning 5 LEFT, 3 IN and 2 UP.



- The MAGNITUDE is written as $|\underline{\boldsymbol{u}}|$ or $|\overrightarrow{AB}|$ and can be calculated using the components:
  $|\underline{\boldsymbol{u}}| = \sqrt{x^2 + y^2 + z^2}$ or $|\overrightarrow{AB}| = \sqrt{x^2 + y^2 + z^2}$, for 2D miss out the z!
- Vectors with the same components are equal vectors, regardless of their starting point, i.e.
  $\overrightarrow{AB} = \begin{pmatrix} -2 \\ 5 \\ 4 \end{pmatrix}, \overrightarrow{CD} = \begin{pmatrix} -2 \\ 5 \\ 4 \end{pmatrix}$ hence $\overrightarrow{AB} = \overrightarrow{CD}$
- Remember direction matters so $\overrightarrow{AB} \neq \overrightarrow{BA}$

Examples:

1. Write down the components of the Vectors below:



$\overrightarrow{AB} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$ $\overrightarrow{DC} = \begin{pmatrix} 3 \\ -6 \end{pmatrix}$ $\overrightarrow{EF} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ $\overrightarrow{GH} = \begin{pmatrix} 10 \\ -2 \end{pmatrix}$

2. Find the magnitude of the Directed Line Segments above.

$|\overrightarrow{AB}| = \sqrt{x^2 + y^2}$
$|\overrightarrow{AB}| = \sqrt{6^2 + 3^2}$
$|\overrightarrow{AB}| = \sqrt{36 + 9}$
$|\overrightarrow{AB}| = \sqrt{45}$
$|\overrightarrow{AB}| = 3\sqrt{5}$

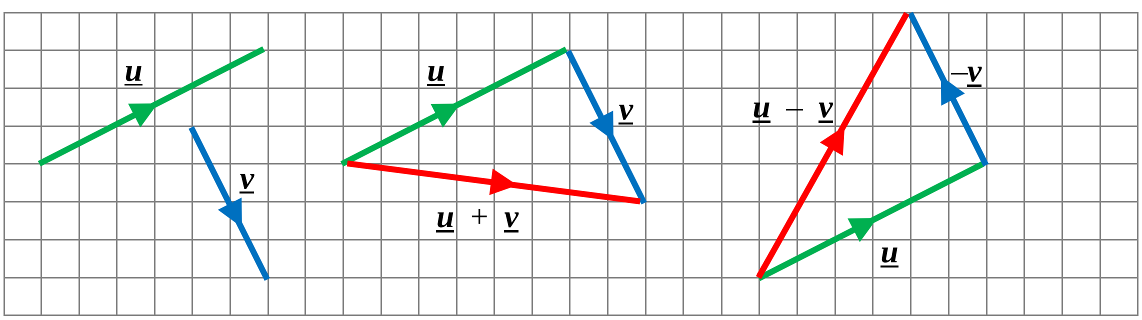
$|\overrightarrow{DC}| = \sqrt{x^2 + y^2}$
$|\overrightarrow{DC}| = \sqrt{3^2 + (-6)^2}$
$|\overrightarrow{DC}| = \sqrt{9 + 36}$
$|\overrightarrow{DC}| = \sqrt{45}$
$|\overrightarrow{DC}| = 3\sqrt{5}$

$|\overrightarrow{EF}| = \sqrt{x^2 + y^2}$
$|\overrightarrow{EF}| = \sqrt{(-1)^2 + 3^2}$
$|\overrightarrow{EF}| = \sqrt{1 + 9}$
$|\overrightarrow{EF}| = \sqrt{10}$

$|\overrightarrow{GH}| = \sqrt{x^2 + y^2}$
$|\overrightarrow{GH}| = \sqrt{10^2 + (-2)^2}$
$|\overrightarrow{GH}| = \sqrt{100 + 4}$
$|\overrightarrow{GH}| = \sqrt{104}$
$|\overrightarrow{GH}| = 2\sqrt{26}$

Adding/Subtracting Vectors & Multiplying by a Scalar:

- When you add or subtract Vectors you produce the **RESULTANT** Vector.
- When subtracting Vectors you actually add the negative Vector, i.e. $\underline{\boldsymbol{u}} - \underline{\boldsymbol{v}} = \underline{\boldsymbol{u}} + (-\underline{\boldsymbol{v}})$
- Vectors are added/subtracted using the Nose to Tail method as shown:



- Their components can be added/subtracted as follows:

If $\underline{\boldsymbol{u}} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\underline{\boldsymbol{v}} = \begin{pmatrix} d \\ e \\ f \end{pmatrix}$ then $\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}} = \begin{pmatrix} a+d \\ b+e \\ c+f \end{pmatrix}$ and $\underline{\boldsymbol{u}} - \underline{\boldsymbol{v}} = \begin{pmatrix} a-d \\ b-e \\ c-f \end{pmatrix}$

- Multiplying a Vector by a Scalar looks like: If $\underline{\boldsymbol{u}} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ then $k\underline{\boldsymbol{u}} = \begin{pmatrix} kx \\ ky \\ kz \end{pmatrix}$
- You cannot multiply a Vector by another Vector.

Examples:

3. For the example above find: **a)** $\underline{u} + \underline{v}$ **(b)** $\underline{u} - \underline{v}$ **(c)** $3\underline{u}$ **(d)** $3\underline{u} + 5\underline{v}$

$\underline{u} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$ and $\underline{v} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$ so:

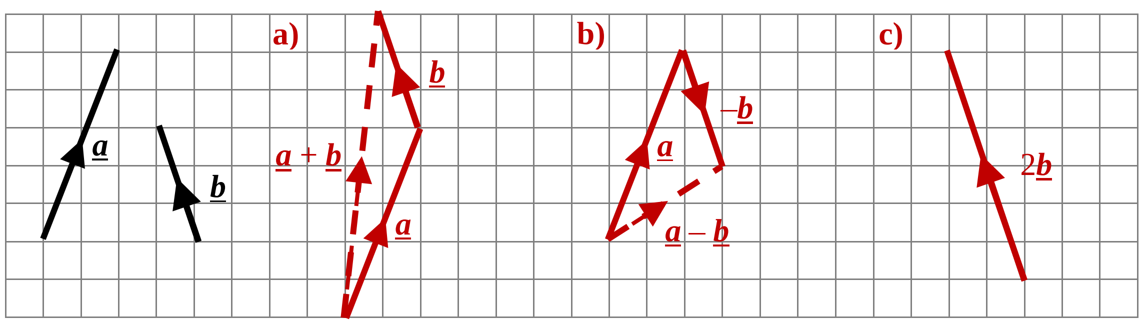
a) $\underline{u} + \underline{v} = \begin{pmatrix} 6 + 2 \\ 3 + (-2) \end{pmatrix} = \begin{pmatrix} 8 \\ -1 \end{pmatrix}$

(b) $\underline{u} - \underline{v} = \begin{pmatrix} 6 - 2 \\ 3 - (-2) \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$

c) $3\underline{u} = 3\begin{pmatrix} 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 18 \\ 9 \end{pmatrix}$

(d) $3\underline{u} + 5\underline{v} = 3\begin{pmatrix} 6 \\ 3 \end{pmatrix} + 5\begin{pmatrix} 2 \\ -4 \end{pmatrix} = \begin{pmatrix} 18 \\ 9 \end{pmatrix} + \begin{pmatrix} 10 \\ -20 \end{pmatrix} = \begin{pmatrix} 28 \\ -11 \end{pmatrix}$

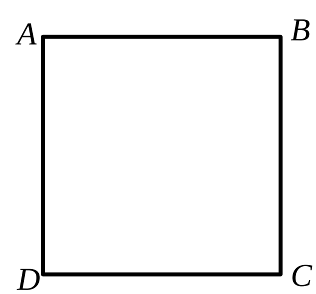
4. For the Vectors below draw the resultant Vector: **a)** $\underline{a} + \underline{b}$ **(b)** $\underline{a} - \underline{b}$ **(c)** $2\underline{b}$



5. State the components of the resultant Vectors above:

a) $\underline{a} + \underline{b} = \begin{pmatrix} 1 \\ 8 \end{pmatrix}$ **(b)** $\underline{a} - \underline{b} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ **(c)** $2\underline{b} = \begin{pmatrix} -2 \\ 6 \end{pmatrix}$

6. In the square opposite name the equivalent vector to:



a) $\overrightarrow{AB}$ **(b)** $\overrightarrow{BC}$ **(c)** $-\overrightarrow{AD}$

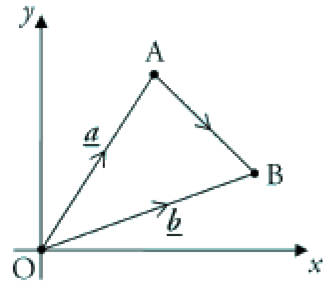
$\overrightarrow{AB} = \overrightarrow{DC}$ $\overrightarrow{BC} = \overrightarrow{AD}$ $-\overrightarrow{AD} = \overrightarrow{DA} = \overrightarrow{CB}$

7. In the same square name the resultant vector when:

a) $\overrightarrow{AB} + \overrightarrow{BC}$ **(b)** $\overrightarrow{BC} - \overrightarrow{AB}$

$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ $\overrightarrow{BC} - \overrightarrow{AB} = \overrightarrow{BD}$

Parallel, Unit and Position Vectors:



- A vector are is PARALLEL to another vector if it is a multiple of the first one, i.e. $\underline{v} = k\underline{u}$ Direction is not important!
- A UNIT VECTOR is a vector with magnitude ONE.
- A POSITION VECTOR is a Vector joing a point, $A(x\ ,y\ ,z)$, to the Origin, i.e. $\overrightarrow{OA} = \underline{a} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$
  - The Vector joining any 2 points, A and B, can be found using the position Vectors: $\overrightarrow{AB} = \underline{b} - \underline{a}$

VERY important

Examples:

8. Prove that the vectors, $\underline{\boldsymbol{a}} = \begin{pmatrix} 2 \\ -4 \\ -8 \end{pmatrix}$ and $\underline{\boldsymbol{b}} = \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix}$ are parallel.

$\underline{\boldsymbol{a}} = \begin{pmatrix} 2 \\ -4 \\ -8 \end{pmatrix} = -2\begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} = -2\underline{b}$ hence $\underline{\boldsymbol{a}}$ is parallel to $\underline{\boldsymbol{b}}$

9. P and Q are the points (3 , –2 , 5) and (–8 , 6 , 4) respectively.
Find the magnitude of vector $\overrightarrow{PQ}$.

$\overrightarrow{PQ} = \underline{q} - \underline{p}$

$\overrightarrow{PQ} = \begin{pmatrix} -8 \\ 6 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}$

$\overrightarrow{PQ} = \begin{pmatrix} -11 \\ 8 \\ -1 \end{pmatrix}$

$|\overrightarrow{PQ}| = \sqrt{x^2 + y^2 + z^2}$

$|\overrightarrow{PQ}| = \sqrt{(-11)^2 + 8^2 + (-1)^2}$

$|\overrightarrow{PQ}| = \sqrt{121 + 64 + 1}$

$|\overrightarrow{PQ}| = \sqrt{186}$

10. Find the components of the unit vector parallel to the vector $\underline{\boldsymbol{a}} = \begin{pmatrix} 12 \\ 0 \\ -5 \end{pmatrix}$

$|\underline{a}| = \sqrt{x^2 + y^2 + z^2}$

$|\underline{a}| = \sqrt{12^2 + 0^2 + (-5)^2}$

$|\underline{a}| = \sqrt{144 + 0 + 25}$

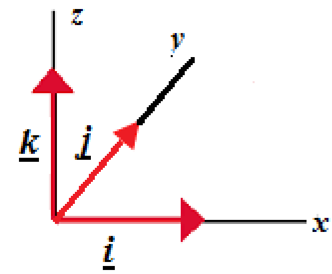
$|\underline{a}| = \sqrt{169}$

$|\underline{a}| = 13$ So the unit vector is $\frac{1}{13}\begin{pmatrix} 12 \\ 0 \\ -5 \end{pmatrix}$

3 Special Unit Vectors:

- Any Vector can be expressed in terms of

  the Unit Vectors $\underline{\boldsymbol{i}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\underline{\boldsymbol{j}} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\underline{\boldsymbol{l}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$



Examples:

11. a) Express the vector, $\underline{\boldsymbol{a}} = \begin{pmatrix} 2 \\ -4 \\ -8 \end{pmatrix}$ in terms of $\underline{\boldsymbol{i}}$, $\underline{\boldsymbol{j}}$, $\underline{\boldsymbol{k}}$: $2\underline{\boldsymbol{i}} - 4\underline{\boldsymbol{j}} - \mathbf{8}\underline{\boldsymbol{k}}$

b) Write the components of the vector $\underline{\boldsymbol{a}} = 5\underline{\boldsymbol{i}} - 3\underline{\boldsymbol{j}} + 2\underline{\boldsymbol{k}}$: $\underline{\boldsymbol{a}} = \begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix}$

c) Write the components of the vector $\underline{\boldsymbol{b}} = -7\underline{\boldsymbol{i}} + 11\underline{\boldsymbol{k}}$: $\underline{\boldsymbol{b}} = \begin{pmatrix} -7 \\ 0 \\ 11 \end{pmatrix}$

d) Calculate the magnitude of the vector $\underline{\boldsymbol{c}} = 5\underline{\boldsymbol{i}} + 2\underline{\boldsymbol{j}} - \underline{\boldsymbol{k}}$

$$|\underline{\boldsymbol{c}}| = \sqrt{x^2 + y^2 + z^2}$$

$$|\underline{\boldsymbol{c}}| = \sqrt{5^2 + 2^2 + (-1)^2}$$

$$|\underline{\boldsymbol{c}}| = \sqrt{25 + 4 + 1}$$

$$|\underline{\boldsymbol{c}}| = \sqrt{30}$$

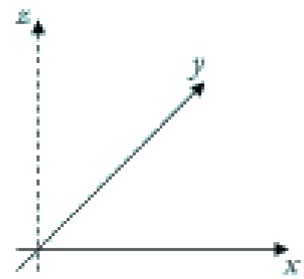
Adapting Known Formulae:

- The following formulae can be used in 3 Dimensions by adding a *z* part to them:
  - Distance Formula: $\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}$
  - Midpoint Formula: $\left(\frac{x_A + x_B}{2} \ , \ \frac{y_A + y_B}{2} \ , \ \frac{z_A + z_B}{2}\right)$
- The Position Vector of the midpoint can be found using the formula: $M = \frac{1}{2}(\underline{\boldsymbol{a}} + \underline{\boldsymbol{b}})$

<u>3D Coordinates:</u>

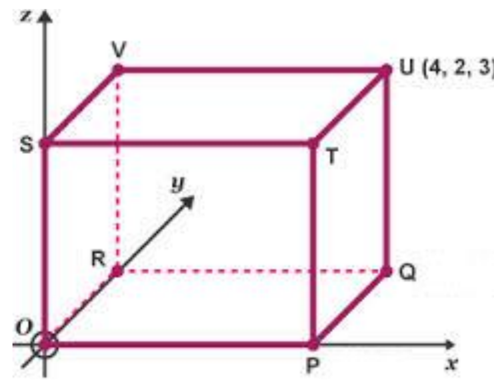
- As seen earlier a point in 3 Dimensions has coordinates in the form: $A(x\ ,y,\ ,z)$

- The x-axis goes from Left to Right, the y-axis goes In & Out and the z-axis goes Up & Down.



<u>Examples:</u>

12. A cuboid is shown in the diagram opposite with Point $U(4\ ,2\ ,3)$.



a) State the coordinates of all the other points:

$O(0\ ,0\ ,0)$ $P(4\ ,0\ ,0)$ $Q(4\ ,2\ ,0)$

$R(0\ ,2\ ,0)$ $S(0\ ,0\ ,3)$ $T(4\ ,0\ ,3)$

$V(0\ ,2\ ,3)$

b) State the components of the **FACE** diagonal $\overrightarrow{VT}$ and the **SPACE** diagonal $\overrightarrow{PV}$ and calculate the magnitudes of these vectors.

$$\overrightarrow{VT} = \underline{t} - \underline{v}$$

$$\overrightarrow{VT} = \begin{pmatrix}4\\0\\3\end{pmatrix} - \begin{pmatrix}0\\2\\3\end{pmatrix}$$

$$\overrightarrow{VT} = \begin{pmatrix}4\\-2\\0\end{pmatrix}$$

$$|\overrightarrow{VT}| = \sqrt{x^2 + y^2 + z^2}$$
$$|\overrightarrow{VT}| = \sqrt{4^2 + (-2)^2 + 0^2}$$
$$|\overrightarrow{VT}| = \sqrt{16 + 4 + 0}$$
$$|\overrightarrow{VT}| = \sqrt{20}$$
$$|\overrightarrow{VT}| = 2\sqrt{5}$$

$$\overrightarrow{PV} = \underline{v} - \underline{p}$$

$$\overrightarrow{PV} = \begin{pmatrix}0\\2\\3\end{pmatrix} - \begin{pmatrix}4\\0\\0\end{pmatrix}$$

$$\overrightarrow{PV} = \begin{pmatrix}-4\\2\\3\end{pmatrix}$$

$$|\overrightarrow{PV}| = \sqrt{x^2 + y^2 + z^2}$$
$$|\overrightarrow{PV}| = \sqrt{(-4)^2 + 2^2 + 3^2}$$
$$|\overrightarrow{PV}| = \sqrt{16 + 4 + 9}$$
$$|\overrightarrow{PV}| = \sqrt{29}$$
$$|\overrightarrow{PV}| = \sqrt{29}$$

c) If $\overrightarrow{OP} = \underline{\boldsymbol{u}}$, $\overrightarrow{OR} = \underline{\boldsymbol{v}}$ and $\overrightarrow{OS} = \underline{\boldsymbol{w}}$ express the vector $\overrightarrow{QS}$ in terms of $\underline{\boldsymbol{u}}$, $\underline{\boldsymbol{v}}$ and $\underline{\boldsymbol{w}}$

$$\overrightarrow{QS} = \overrightarrow{QP} + \overrightarrow{PT} + \overrightarrow{TS}$$

$$\overrightarrow{QS} = -\overrightarrow{OR} + \overrightarrow{OS} - \overrightarrow{OP}$$

$$\overrightarrow{QS} = -\underline{\boldsymbol{v}} + \underline{\boldsymbol{w}} - \underline{\boldsymbol{u}}$$

d) Prove that $\angle PRV$ is a right angle.

$\overrightarrow{PR} = \underline{r} - \underline{p}$

$\overrightarrow{PR} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$

$\overrightarrow{PR} = \begin{pmatrix} -4 \\ 2 \\ 0 \end{pmatrix}$

$\overrightarrow{RV} = \underline{v} - \underline{r}$

$\overrightarrow{RV} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$

$\overrightarrow{RV} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$

$\overrightarrow{PV} = \underline{v} - \underline{p}$

$\overrightarrow{PV} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$

$\overrightarrow{PV} = \begin{pmatrix} -4 \\ 2 \\ 3 \end{pmatrix}$

$|\overrightarrow{PR}| = \sqrt{x^2 + y^2 + z^2}$

$|\overrightarrow{PR}| = \sqrt{(-4)^2 + 2^2 + 0^2}$

$|\overrightarrow{PR}| = \sqrt{16 + 4 + 0}$

$|\overrightarrow{PR}| = \sqrt{20}$

$|\overrightarrow{RV}| = \sqrt{x^2 + y^2 + z^2}$

$|\overrightarrow{RV}| = \sqrt{0^2 + 0^2 + 3^2}$

$|\overrightarrow{RV}| = \sqrt{0 + 0 + 9}$

$|\overrightarrow{RV}| = \sqrt{9}$

$|\overrightarrow{RV}| = 3$

$|\overrightarrow{PV}| = \sqrt{x^2 + y^2 + z^2}$

$|\overrightarrow{PV}| = \sqrt{(-4)^2 + 2^2 + 3^2}$

$|\overrightarrow{PV}| = \sqrt{16 + 4 + 9}$

$|\overrightarrow{PV}| = \sqrt{29}$

Now use the CONVERSE of PYTHAGORIAS from National 5:

$|\overrightarrow{PR}| = \sqrt{20} \quad \Longrightarrow \quad |\overrightarrow{PR}|^2 = (\sqrt{20})^2 = 20$

$|\overrightarrow{RV}| = 3 \quad \Longrightarrow \quad |\overrightarrow{RV}|^2 = 3^2 = 9 \quad$ so $|\overrightarrow{PR}|^2 + |\overrightarrow{RV}|^2 = 20 + 9 = 29$

$|\overrightarrow{PV}| = \sqrt{29} \quad \Longrightarrow \quad |\overrightarrow{PV}|^2 = (\sqrt{29})^2 = 29$

Since $|\overrightarrow{PR}|^2 + |\overrightarrow{RV}|^2 = |\overrightarrow{PV}|^2$ the angle $\angle PRV$ is a right angle.

We will see shortly a quicker way of proving that 2 Vectors are Perpendicular!

e) Find the coordinates of the midpoint of $\overrightarrow{ST}$

Midpoint Formula

$M = \left(\frac{x_A + x_B}{2}, \frac{y_A + y_B}{2}, \frac{z_A + z_B}{2}\right)$

$M = \left(\frac{0+4}{2}, \frac{0+0}{2}, \frac{3+3}{2}\right)$

$M = \left(\frac{4}{2}, \frac{0}{2}, \frac{6}{2}\right)$

$M = (2, 0, 3)$

OR

Position Vector Midpoint

$M = \frac{1}{2}\left[\begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}\right]$

$M = \frac{1}{2}\begin{pmatrix} 4 \\ 0 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$ ← **Vector Components**

$M = (2, 0, 3)$

↑ **Coordinates**

Collinearity:

- As seen earlier in the STRAIGHT LINE topic, 3 points are said to be COLLINEAR , i.e. lie on a straight line if the gradient between pairs of points are equal (PARALLEL) and if one of the points is common to both Gradients.
- The same is true for 3D Vectors, remember for Parallel Vectors: $\underline{\boldsymbol{v}} = k\underline{\boldsymbol{u}}$
- We cannot find Gradients in 3D, but $\underline{\boldsymbol{v}} = k\underline{\boldsymbol{u}}$ is the equivalent.

Examples:

13. a) Prove that the points $A(0 , –2 , 5)$, $B(4 , 2 , 3)$ and $C(10 , 8 , 0)$ are collinear.

$$\overrightarrow{AB} = \underline{b} - \underline{a} \qquad \overrightarrow{BC} = \underline{c} - \underline{b}$$

$$\overrightarrow{AB} = \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 0 \\ -2 \\ 5 \end{pmatrix} \qquad \overrightarrow{BC} = \begin{pmatrix} 10 \\ 8 \\ 0 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}$$

$$\overrightarrow{AB} = \begin{pmatrix} 4 \\ 4 \\ -2 \end{pmatrix} \qquad \overrightarrow{BC} = \begin{pmatrix} 6 \\ 6 \\ -3 \end{pmatrix}$$

$$\overrightarrow{AB} = 2\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \qquad \overrightarrow{BC} = 3\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$$

so $3\overrightarrow{AB} = 2\overrightarrow{BC}$

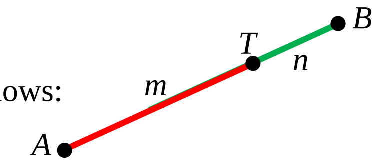
so $\overrightarrow{AB} = \frac{2}{3}\overrightarrow{BC}$, Therefore $\overrightarrow{AB}$ is parallel to $\overrightarrow{BC}$ and since B is a common point, A, B & C are collinear.

Statement MUST be written

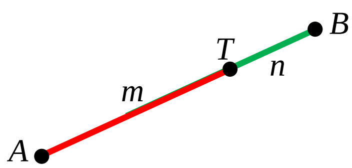
b) Find the ratio $\overrightarrow{AB} : \overrightarrow{BC}$

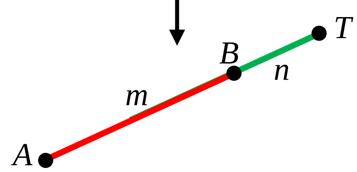
$\overrightarrow{AB} = \frac{2}{3}\overrightarrow{BC} \implies \frac{\overrightarrow{AB}}{\overrightarrow{BC}} = \frac{2}{3}$, so $\overrightarrow{AB} : \overrightarrow{BC}$ is $2 : 3$

Dividing Lines:



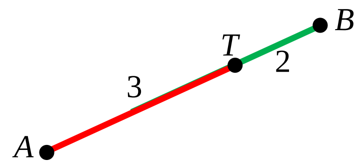
- A point can split a line joing 2 other points in a given ratio as follows:
- So T splits the line AB in the ratio $m : n$
- If we know the coordinates of 2 points we can find the third point in 2 ways:
  - Algebraicly using the fact that $\overrightarrow{AB} = \underline{b} - \underline{a}$
  - Or Using the SECTION FORMULA: $\underline{t} = \frac{n}{m+n}\underline{\boldsymbol{a}} + \frac{m}{m+n}\underline{\boldsymbol{b}}$
- The point T can either split the line AB **INTERNALLY** or **EXTERNALLY** as shown below:





Examples:

14. The point T divides the line AB in the ratio 3 : 2.
For the coordinates $A(-4, 5, 1)$ & $B(-24, -10, 26)$
find the coordinates of the point T.



Section Formula

$$\underline{t} = \frac{n}{m+n}\underline{\boldsymbol{a}} + \frac{m}{m+n}\underline{\boldsymbol{b}}$$

$$\underline{t} = \frac{2}{3+2}\begin{pmatrix}-4\\5\\1\end{pmatrix} + \frac{3}{3+2}\begin{pmatrix}-24\\-10\\26\end{pmatrix}$$

$$\underline{t} = \frac{2}{5}\begin{pmatrix}-4\\5\\1\end{pmatrix} + \frac{3}{5}\begin{pmatrix}-24\\-10\\26\end{pmatrix}$$

$$\underline{t} = \begin{pmatrix}-{}^{8}/_{5}\\2\\{}^{2}/_{5}\end{pmatrix} + \begin{pmatrix}-{}^{72}/_{5}\\-6\\{}^{78}/_{5}\end{pmatrix}$$

$$\underline{t} = \begin{pmatrix}-{}^{80}/_{5}\\-4\\{}^{80}/_{5}\end{pmatrix} = \begin{pmatrix}-16\\-4\\16\end{pmatrix}$$

So T (–16 , –4 , 16)

Algebraicly

$$2\overrightarrow{AT} = 3\overrightarrow{TB} \quad \Rightarrow \quad 2(\underline{t} - \underline{a}) = 3(\underline{b} - \underline{t})$$

$$\Rightarrow \quad 2\underline{t} - 2\underline{a} = 3\underline{b} - 3\underline{t}$$

$$\Rightarrow \quad 5\underline{t} = 3\underline{b} + 2\underline{a}$$

$$5\underline{t} = 3\begin{pmatrix}-24\\-10\\26\end{pmatrix} + 2\begin{pmatrix}-4\\5\\1\end{pmatrix}$$

$$5\underline{t} = \begin{pmatrix}-72\\-30\\78\end{pmatrix} + \begin{pmatrix}-2\\10\\2\end{pmatrix}$$

$$5\underline{t} = \begin{pmatrix}-80\\-20\\80\end{pmatrix}$$

$$\underline{t} = \begin{pmatrix}-16\\-4\\16\end{pmatrix}$$

So T (–16 , –4 , 16)

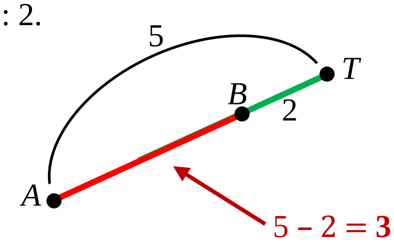
Components

Coordinates

15. The point T divides the line AB **externally** in the ratio 5 : 2.

For the coordinates $A(3, -1, 4)$ & $B(6, -4, 1)$

find the coordinates of the point T.



$$2\overrightarrow{AB} = 3\overrightarrow{BT} \quad \Longrightarrow \quad 2(\underline{b} - \underline{a}) = 3(\underline{t} - \underline{b})$$

$$\Longrightarrow \quad 2\underline{b} - 2\underline{a} = 3\underline{t} - 3\underline{b}$$

$$\Longrightarrow \quad 3\underline{t} = 5\underline{b} - 2\underline{a}$$

$$3\underline{t} = 5\begin{pmatrix} 6 \\ -4 \\ 1 \end{pmatrix} - 2\begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$$

$$3\underline{t} = \begin{pmatrix} 30 \\ -20 \\ 5 \end{pmatrix} - \begin{pmatrix} 6 \\ -2 \\ 8 \end{pmatrix}$$

$$3\underline{t} = \begin{pmatrix} 24 \\ -18 \\ -3 \end{pmatrix}$$

$$\underline{t} = \begin{pmatrix} 8 \\ -6 \\ -1 \end{pmatrix}$$

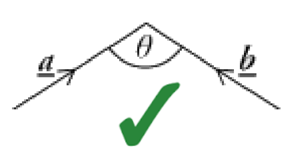
So T (8 , -6 , -1)

Scalar Product:

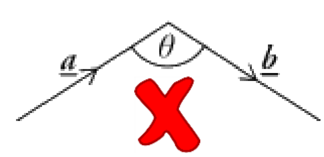
- The SCALAR PRODUCT is as close to multiplying vectors together as we can get!
- The Scalar Product is sometimes known as the DOT Product as it is written in the form: $\underline{\boldsymbol{a}}.\underline{\boldsymbol{b}}$
- The Scalar Product can be calculated in 2 ways:
  - The component form: $\underline{a}.\underline{b} = a_1 \times b_1 + a_2 \times b_2 + a_3 \times b_3$
  - The Modulus form: $\underline{a}.\underline{b} = |\underline{a}||\underline{b}|\cos\theta$

    Where θ is the angle betweem the vectors $\underline{a}$ and $\underline{b}$
- To use the Modulus form above both vectors must either be moving away from or towards the angle:

Both moving away from angle

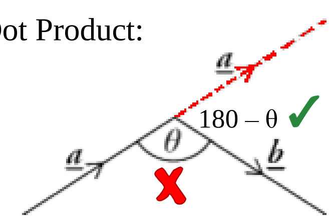


Both moving towards the angle



$\underline{\boldsymbol{a}}$ is moving towards and $\underline{\boldsymbol{b}}$ is moving away from the angle.

- We can reaarange the 3rd diagram to allow us to find the Dot Product:



Examples:

16. Find $\underline{a}.\underline{b}$ given that $\underline{\boldsymbol{a}} = \begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix}$ and $\underline{\boldsymbol{b}} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$

$$\underline{a}.\underline{b} = a_1 \times b_1 + a_2 \times b_2 + a_3 \times b_3$$
$$\underline{a}.\underline{b} = 5 \times 3 + (-3) \times 4 + 2 \times (-1)$$
$$\underline{a}.\underline{b} = 15 + (-12) + (-2) = 1$$

17. Find $\overrightarrow{PQ}.\overrightarrow{PR}$ given that $P(2, -4, 9)$, $Q(3, -1, 4)$ & $R(6, -4, 1)$

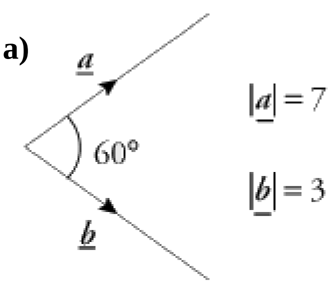
$$\overrightarrow{PQ} = \underline{q} - \underline{p}$$
$$\overrightarrow{PQ} = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ -4 \\ 9 \end{pmatrix}$$
$$\overrightarrow{PQ} = \begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix}$$

$$\overrightarrow{PR} = \underline{r} - \underline{p}$$
$$\overrightarrow{PR} = \begin{pmatrix} 6 \\ -4 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ -4 \\ 9 \end{pmatrix}$$
$$\overrightarrow{PR} = \begin{pmatrix} 4 \\ 0 \\ -8 \end{pmatrix}$$

$$\overrightarrow{PQ}.\overrightarrow{PR} = a_1 \times b_1 + a_2 \times b_2 + a_3 \times b_3$$
$$\overrightarrow{PQ}.\overrightarrow{PR} = 1 \times 4 + 3 \times 0 + (-5) \times (-8)$$
$$\overrightarrow{PQ}.\overrightarrow{PR} = 4 + 0 + 40 = 44$$

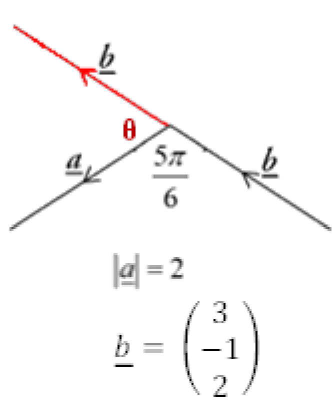
18. For each diagram below calculate $\underline{a}.\underline{b}$:

a)



$$\underline{a}.\underline{b} = |\underline{a}||\underline{b}| \cos\theta$$
$$\underline{a}.\underline{b} = 7 \times 3 \times \cos 60$$
$$\underline{a}.\underline{b} = 21 \times 0.5$$
$$\underline{a}.\underline{b} = 10.5$$

b)



$$\frac{5\pi}{6} = 150°$$
$$\theta = 180 - 150$$
$$\theta = 30$$

$$|\underline{\boldsymbol{b}}| = \sqrt{x^2 + y^2 + z^2}$$
$$|\underline{\boldsymbol{b}}| = \sqrt{3^2 + (-1)^2 + 2^2}$$
$$|\underline{\boldsymbol{b}}| = \sqrt{9 + 1 + 4}$$
$$|\underline{\boldsymbol{b}}| = \sqrt{14}$$

$$\underline{a}.\underline{b} = |\underline{a}||\underline{b}| \cos\theta$$
$$\underline{a}.\underline{b} = 2 \times \sqrt{14} \times \cos 30$$
$$\underline{a}.\underline{b} = 2\sqrt{14} \times \frac{\sqrt{3}}{2}$$
$$\underline{a}.\underline{b} = \sqrt{14} \times \sqrt{3} = \sqrt{42}$$

Angle Between 2 Vectors:

- We can reaarange the Scalar Product to allow us to be able to find the Angle inbetween the 2 vectors:

$$\cos\theta = \frac{\underline{a}.\underline{b}}{|\underline{a}||\underline{b}|} = \frac{a_1 \times b_1 + a_2 \times b_2 + a_3 \times b_3}{|\underline{a}||\underline{b}|}$$

Examples:

19. Calculate the angle, θ, between the 2 vectors, $\underline{\boldsymbol{m}} = 3\underline{\boldsymbol{i}} + 4\underline{\boldsymbol{j}} - 2\underline{\boldsymbol{k}}$ and $\underline{\boldsymbol{n}} = 4\underline{\boldsymbol{i}} + \underline{\boldsymbol{j}} + 3\underline{\boldsymbol{k}}$

$\underline{\boldsymbol{m}} = 3\underline{\boldsymbol{i}} + 4\underline{\boldsymbol{j}} - 2\underline{\boldsymbol{k}}$

$\underline{\boldsymbol{m}} = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}$

$|\underline{\boldsymbol{m}}| = \sqrt{x^2 + y^2 + z^2}$

$|\underline{\boldsymbol{m}}| = \sqrt{3^2 + 4^2 + (-2)^2}$

$|\underline{\boldsymbol{m}}| = \sqrt{9 + 16 + 4}$

$|\underline{\boldsymbol{m}}| = \sqrt{29}$

$\underline{\boldsymbol{n}} = 4\underline{\boldsymbol{i}} + \underline{\boldsymbol{j}} + 3\underline{\boldsymbol{k}}$

$\underline{\boldsymbol{n}} = \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}$

$|\underline{\boldsymbol{n}}| = \sqrt{x^2 + y^2 + z^2}$

$|\underline{\boldsymbol{n}}| = \sqrt{4^2 + 1^2 + 3^2}$

$|\underline{\boldsymbol{n}}| = \sqrt{16 + 1 + 9}$

$|\underline{\boldsymbol{n}}| = \sqrt{26}$

$\underline{m}.\underline{n} = a_1 \times b_1 + a_2 \times b_2 + a_3 \times b_3$

$\underline{m}.\underline{n} = 3 \times 4 + 4 \times 1 + (-2) \times 3$

$\underline{m}.\underline{n} = 12 + 4 + (-6) = 10$

$\cos\theta = \frac{\underline{m}.\underline{n}}{|\underline{m}||\underline{n}|}$

$\cos\theta = \frac{10}{\sqrt{29} \times \sqrt{26}}$

$\theta = \cos^{-1} 0.364 \ldots$

$\theta = 68.64°$

20. Find *LDEF* when *D*(2 , 5 , 1) , *E*(–3 , 3 , 4) & *F*(1 , –7 , 2)

As both vectors MUST travel in the same direction start from the outsides of the angle's name and work in!

$\overrightarrow{DE} = \underline{e} - \underline{d}$

$\overrightarrow{DE} = \begin{pmatrix} -3 \\ 3 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}$

$\overrightarrow{DE} = \begin{pmatrix} -5 \\ -2 \\ 3 \end{pmatrix} = \underline{a}$

$|\underline{\boldsymbol{a}}| = \sqrt{x^2 + y^2 + z^2}$

$|\underline{\boldsymbol{a}}| = \sqrt{(-5)^2 + (-2)^2 + 3^2}$

$|\underline{\boldsymbol{a}}| = \sqrt{25 + 4 + 9}$

$|\underline{\boldsymbol{a}}| = \sqrt{38}$

$\overrightarrow{FE} = \underline{e} - \underline{f}$

$\overrightarrow{FE} = \begin{pmatrix} -3 \\ 3 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ -7 \\ 2 \end{pmatrix}$

$\overrightarrow{FE} = \begin{pmatrix} -4 \\ 10 \\ 2 \end{pmatrix} = \underline{b}$

$|\underline{\boldsymbol{b}}| = \sqrt{x^2 + y^2 + z^2}$

$|\underline{\boldsymbol{b}}| = \sqrt{(-4)^2 + 10^2 + 2^2}$

$|\underline{\boldsymbol{b}}| = \sqrt{16 + 100 + 4}$

$|\underline{\boldsymbol{b}}| = \sqrt{120}$

$\underline{a}.\underline{b} = a_1 \times b_1 + a_2 \times b_2 + a_3 \times b_3$

$\underline{a}.\underline{b} = (-5) \times (-4) + (-2) \times 10 + 3 \times 2$

$\underline{a}.\underline{b} = 20 + (-20) + 6 = 6$

$cos\,\theta = \frac{a.b}{|\underline{a}||\underline{b}|}$

$cos\,\theta = \frac{6}{\sqrt{39} \times \sqrt{120}}$

$\theta = \cos^{-1} 0.0877 \ldots$

$\theta = 84.97°$

Perpendicular Vectors:

- If vectors $\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{b}}$ are perpendicular then $\underline{a}.\underline{b} = 0$

Examples:

21. Show that vectors $\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{b}}$ are perpendicular when $\underline{\boldsymbol{a}} = 3\underline{\boldsymbol{i}} - 5\underline{\boldsymbol{j}} + \underline{\boldsymbol{k}}$ and $\underline{\boldsymbol{b}} = 4\underline{\boldsymbol{i}} + 3\underline{\boldsymbol{j}} + 3\underline{\boldsymbol{k}}$

$\underline{\boldsymbol{a}} = 3\underline{\boldsymbol{i}} - 5\underline{\boldsymbol{j}} + \underline{\boldsymbol{k}}$ $\qquad$ $\underline{\boldsymbol{b}} = 4\underline{\boldsymbol{i}} + 3\underline{\boldsymbol{j}} + 3\underline{\boldsymbol{k}}$

$\underline{\boldsymbol{a}} = \begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix}$ $\qquad$ $\underline{\boldsymbol{b}} = \begin{pmatrix} 4 \\ 3 \\ 3 \end{pmatrix}$

$\underline{a}.\underline{b} = a_1 \times b_1 + a_2 \times b_2 + a_3 \times b_3$

$\underline{a}.\underline{b} = 3 \times 4 + (-5) \times 3 + 1 \times 3$

$\underline{a}.\underline{b} = 12 + (-15) + 3 = 0$ Since $\underline{a}.\underline{b} = 0$ $\underline{\boldsymbol{a}}$ is perpendicular to $\underline{\boldsymbol{b}}$.

22. Find the value of k if the direct line segments $\overrightarrow{DE} = \begin{pmatrix} 3 \\ -2 \\ k \end{pmatrix}$ and $\overrightarrow{FG} = \begin{pmatrix} -4 \\ -5 \\ 2 \end{pmatrix}$ are perpendicular.

Since $\overrightarrow{DE}$ is perpendicular to $\overrightarrow{FG}$

$$\overrightarrow{DE}.\overrightarrow{FG} = 0$$

$$a_1 \times b_1 + a_2 \times b_2 + a_3 \times b_3 = 0$$

$$3 \times (-4) + (-2) \times (-5) + k \times 2 = 0$$

$$-12 + 15 + 2k = 0$$

$$3 + 2k = 0$$

$$2k = -3$$

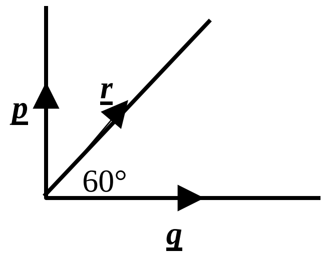
$$k = -1.5$$

Scalar Product Properties:

- For Vectors vectors $\underline{a}$, $\underline{b}$ and $\underline{c}$ the following 3 properties exist:
    - $\underline{a}.\underline{b} = \underline{b}.\underline{a}$
    - $\underline{a}.(\underline{b} + \underline{c}) = \underline{a}.\underline{b} + \underline{a}.\underline{c}$
    - $\underline{a}.\underline{a} = |\underline{a}|^2$

Examples:

23. Given that vectors $\underline{p}$ and $\underline{q}$ are perpendicular and $|\underline{p}| = 2$, $|\underline{q}| = 3$, $|\underline{r}| = \sqrt{3}$ calculate the exact value of $\underline{p} \cdot (\underline{q} + \underline{r})$



Angle between vectors $\underline{p}$ and $\underline{r}$ is: 90 – 60 = 30°

$$\underline{p} \cdot (\underline{q} + \underline{r}) = \underline{p} \cdot \underline{q} + \underline{p} \cdot \underline{r}$$

$$= 0 + |\underline{p}|\,|\underline{r}| \cos\theta$$

Since vectors $\underline{p}$ and $\underline{q}$ are perpendicular. $\underline{p} \cdot \underline{q} = 0$

$$= 2 \times \sqrt{3} \times \cos 30$$

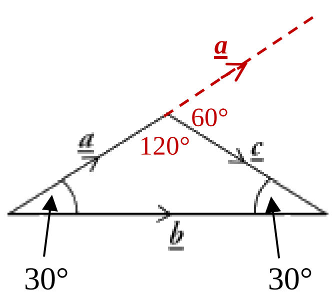
$$= 2\sqrt{3} \times \frac{\sqrt{3}}{2}$$

$$= \frac{2 \times \sqrt{3} \times \sqrt{3}}{2}$$

$$= \sqrt{9}$$

$$= 3$$

24. For the diagram opposite and $|\underline{a}| = 4$, $|\underline{b}| = 4\sqrt{3}$ calculate the exact value of $\underline{a} \cdot (\underline{a} + \underline{b} + \underline{c})$



$$\underline{a} \cdot (\underline{a} + \underline{b} + \underline{c}) = \underline{a} \cdot \underline{a} + \underline{a} \cdot \underline{b} + \underline{a} \cdot \underline{c}$$

$$= 4^2 + 4 \times 4\sqrt{3} \times \cos 30 + 4 \times 4 \times \cos 60$$

$$= 16 + 16\sqrt{3} \times \frac{\sqrt{3}}{2} + 16 \times \frac{1}{2}$$

$$= 16 + \frac{16\sqrt{9}}{2} + 8$$

$$= 16 + 24 + 8$$

$$= 48$$