Wave Function

SPTA Mathematics – Higher Notes

- A Trig expression in the form $a\cos x° + b\sin x°$ can be written in the form: $k\cos(x \pm \alpha)°$ or $k\sin(x \pm \alpha)°$

  $k = \sqrt{a^2 + b^2}$ $\quad Tan\alpha° = \dfrac{k\sin\alpha°}{k\cos\alpha°}$

- You can convert these expressions by following the steps below:
  1. Expand the expression $k\cos(x \pm \alpha)°$ or $k\sin(x \pm \alpha)°$
  2. Compare with the given expression to write down $k\cos\alpha°$ and $k\sin\alpha°$
  3. Use the CAST diagram to find the quadrant for angle $\alpha°$.
  4. Use the 2 formulae above to calculate k and α.
  5. State the expression in the form asked for.

Examples:

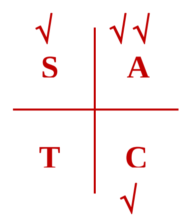
1. Write $5\cos x + 12\sin x$ in the form $k\cos(x - \alpha)°$ where $0 \leq \alpha \leq 360°$

$k\cos(x - \alpha)° = k\cos x\cos\alpha + k\sin x\sin\alpha$ ← **Step 1**

$5\cos x + 12\sin x$ ← **Step 2**

so $k\cos\alpha = 5$ and $k\sin\alpha = 12$ ← **Step 2**

Step 3



Hence α is in 1st quadrant

Step 4

$k = \sqrt{a^2 + b^2}$ $\qquad Tan\alpha = \dfrac{k\sin\alpha}{k\cos\alpha}$

$k = \sqrt{5^2 + 12^2}$ $\qquad Tan\alpha = \dfrac{12}{5}$

$k = \sqrt{25 + 144}$ $\qquad \alpha = \tan^{-1} 2.4$

$k = \sqrt{169}$ $\qquad \alpha = 67.38°$

$k = 13$

So $5\cos x + 12\sin x = 13\cos(x - 67.38)°$ ← **Step 5**

2. Write $5\cos x - 3\sin x$ in the form $k\cos(x + \alpha)°$ where $0 \le \alpha \le 360°$

$k\cos(x + \alpha)° \quad = \quad k\cos x\cos\alpha - k\sin x\sin\alpha$

$\qquad\qquad\qquad\qquad 5\cos x - 3\sin x$

so $\quad k\cos\alpha = 5 \quad$ and $\quad k\sin\alpha = 3$

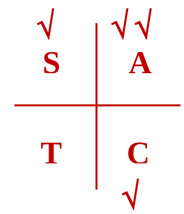
$k = \sqrt{a^2 + b^2} \qquad\qquad Tan\alpha = \frac{ksin\alpha}{kcos\alpha}$

$k = \sqrt{5^2 + 3^2} \qquad\qquad Tan\alpha = \frac{3}{5}$

$k = \sqrt{25 + 9} \qquad\qquad \alpha = \tan^{-1} 0.6$

$k = \sqrt{34} \qquad\qquad \alpha = 30.96°$

$\alpha = 30.96°$



Hence α is in 1st quadrant

So $\quad 5\cos x - 3\sin x = \sqrt{34}\cos(x + 30.96)°$

3. Write $4\cos x - 5\sin x$ in the form $k\sin(x - \alpha)°$ where $0 \le \alpha \le 360°$

$k\sin(x - \alpha)° \quad = \quad k\sin x\cos\alpha - k\cos x\sin\alpha$

$\qquad\qquad\qquad\qquad -5\sin x + 4\cos x$

Rearrange to match the expansion.

so $\quad k\cos\alpha = -5 \quad$ and $\quad k\sin\alpha = 4$

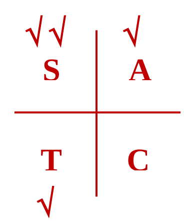
$k = \sqrt{a^2 + b^2} \qquad\qquad Tan\alpha = \frac{ksin\alpha}{kcos\alpha}$

$k = \sqrt{(-5)^2 + 4^2} \qquad\qquad Tan\alpha = \frac{4}{-5}$

$k = \sqrt{25 + 16} \qquad\qquad \alpha = \tan^{-1}(-0.8)$

$k = \sqrt{41} \qquad\qquad \alpha = 180 - 38.66°$

$\alpha = 141.34°$



Hence α is in 2nd quadrant

So $\quad 4\cos x - 5\sin x = \sqrt{41}\sin(x - 141.34)°$

Never put a negative Trig ratio into your calculator – use the CAST diagram!!

Sometimes there is a multiple x term such as $2x$ or $3x$, but this makes no difference to the process.

4. Write $-\sqrt{3}\cos 2x - 2\sin 2x$ in the form $k\sin(2x + \alpha)°$ where $0 \leq \alpha \leq 360°$

$$k\sin(2x+\alpha)° = k\sin 2x\cos\alpha + k\cos 2x\sin\alpha$$
$$-2\sin 2x - \sqrt{3}\cos 2x$$

so $k\cos\alpha = -2$ and $k\sin\alpha = -\sqrt{3}$

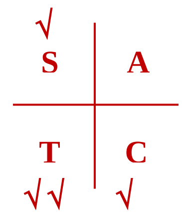
$$k = \sqrt{a^2 + b^2} \qquad Tan\alpha = \frac{k\sin\alpha}{k\cos\alpha}$$

$$k = \sqrt{(-\sqrt{3})^2 + (-2)^2} \qquad Tan\alpha = \frac{-\sqrt{3}}{-2}$$

$$k = \sqrt{3+4} \qquad \alpha = \tan^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$k = \sqrt{7} \qquad \alpha = 180 + 40.89°$$

$$\alpha = 220.89°$$



Hence α is in 3rd quadrant

So $-\sqrt{3}\cos 2x - 2\sin 2x = \sqrt{7}\sin(2x + 220.89)°$

Maximum and Minimum Values:

- Remember that $y = \sin x°$ and $y = \cos x°$ have a Maximum of 1 and a Minimum of -1.
- We can find the Max/Min values of expressions in the form $a\cos x° + b\sin x°$ by firstly converting them into a single trig expression in the form: $y = k\cos(x \pm \alpha)°$ or $y = k\sin(x \pm \alpha)°$.

Examples:

5. a) Write $4\sin x - \cos x$ in the form $k\cos(x - \alpha)°$ where $0 \leq \alpha \leq 360°$

$$k\cos(x-\alpha)° = k\cos x\cos\alpha + k\sin x\sin\alpha$$
$$-\cos x + 4\sin x$$

Choos an expansion that matches the expression

so $k\cos\alpha = -1$ and $k\sin\alpha = 4$

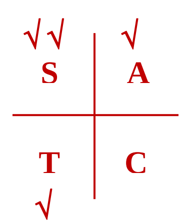
$$k = \sqrt{a^2 + b^2} \qquad Tan\alpha = \frac{k\sin\alpha}{k\cos\alpha}$$

$$k = \sqrt{(-1)^2 + 4^2} \qquad Tan\alpha = \frac{4}{-1}$$

$$k = \sqrt{1+16} \qquad \alpha = \tan^{-1}(-4)$$

$$k = \sqrt{17} \qquad \alpha = 180 - 75.96°$$

$$\alpha = 104.04°$$



Hence α is in 2nd quadrant

So $4\sin x - \cos x = \sqrt{17}\cos(x - 104.04)°$

b) State the maximum and minimum values and the corresponding values of x.

So Max $= \sqrt{17}$ and Min $= -\sqrt{17}$

Maximum occurs when: $\cos(x - 104.04)° = 1$

$x - 104.04° = \cos^{-1}(1)$

$x - 104.04° = 0°$ or $360°$

$x = 104.04°$ or ~~464.04°~~

$x = 104.04°$

No need for a calculator here, find the values from the graph.

Minimum occurs when: $\cos(x - 104.04)° = -1$

$x - 104.04° = \cos^{-1}(-1)$

$x - 104.04° = 180°$

$x = 284.04°$

Solving Equations:

- The wave function can be used to solve equations in the form acos(nx)° + bsin(nx)° by first converting it into a single trig expression in the form: $y = k\cos(x \pm \alpha)°$ or $y = k\sin(x \pm \alpha)°$.

Examples:

6. Solve $5\cos x + \sin x = 2$ where $0 \leq \alpha \leq 180°$

$k\cos(x - \alpha)° = k\cos x\cos\alpha + k\sin x\sin\alpha$

$5\cos x + \sin x$

so $k\cos\alpha = 5$ and $k\sin\alpha = 1$

$k = \sqrt{a^2 + b^2}$

$k = \sqrt{5^2 + 1^2}$

$k = \sqrt{25 + 1}$

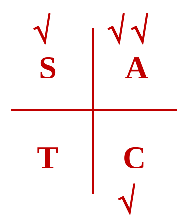
$k = \sqrt{26}$

$Tan\alpha = \frac{k\sin\alpha}{k\cos\alpha}$

$Tan\alpha = \frac{1}{5}$

$\alpha = \tan^{-1}(0.2)$

$\alpha = 11.31°$



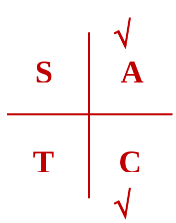
Hence α is in 1st quadrant

So $5\cos x + \sin x = \sqrt{26}\cos(x - 11.31)°$

So $5\cos x + \sin x = 2 \implies \sqrt{26}\cos(x - 11.31)° = 2$

$\cos(x - 11.31)° = \frac{2}{\sqrt{26}}$

2 solutions per x, use the CAST diagram to find the related angle.



$x - 11.31° = \cos^{-1}\left(\frac{2}{\sqrt{26}}\right)$

$x - 11.31° = 66.91°$ or $360 - 66.91°$

$x - 11.31° = 66.91°$ or $293.09°$

$x = 78.22°$ or ~~304.40°~~

Outwith range

$x = 78.22°$

7. Solve $2\cos 2x - 3\sin 2x = 1$ where $0 \le \alpha \le 2\pi$

$$k\cos(2x + \alpha)° = k\cos 2x\cos\alpha - k\sin 2x\sin\alpha$$
$$2\cos 2x - 3\sin 2x$$

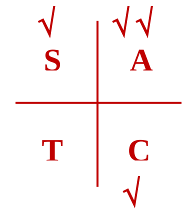
so $k\cos\alpha = 2$ and $k\sin\alpha = 3$

$k = \sqrt{a^2 + b^2}$ $\quad Tan\alpha = \frac{k\sin\alpha}{k\cos\alpha}$

$k = \sqrt{2^2 + 3^2}$ $\quad Tan\alpha = \frac{3}{2}$

$k = \sqrt{4 + 9}$ $\quad \alpha = \tan^{-1}(1.5)$

$k = \sqrt{13}$ $\quad \alpha = 56.31°$

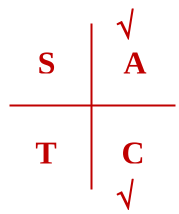


Hence α is in 1st quadrant

So $2\cos 2x - 3\sin 2x = \sqrt{13}\cos(2x + 56.31)°$

So $2\cos 2x - 3\sin 2x = 1 \Longrightarrow \sqrt{13}\cos(2x + 56.31)° = 1$

$$\cos(2x - 56.31)° = \frac{1}{\sqrt{13}}$$



2 solutions per *x*, since 2x there are 4 solutions

$$2x - 56.31° = \cos^{-1}\left(\frac{1}{\sqrt{13}}\right)$$

$$2x - 56.31° = 73.9° \text{ or } 360 - 73.9°$$

$$2x - 56.31° = 73.9° \text{ or } 286.1° \text{ or } 433.9° \text{ or } 646.1°$$

$$2x = 130.21°, 342.41°, 490.21°, 702.41°$$

$$x = 65.11°, 171.21°, 245.11°, 351.21°$$

$$x = 1.14, 2.99, 4.28, 6.13 \text{ radians}$$

Convert to RADIANS by multiplying by π and dividing by 180. Only exact value radiamns are left as a fraction of π

Sketching Graphs:

- You may also be asked to Draw the Graph of an equation in the form $y = a\cos(nx)° + b\sin(nx)°$ by first converting it into a single trig expression.
- We saw earlier in the course how to sketch Trig Graphs.

Examples:

8. Draw the grap of the trig equation $y = \sqrt{3}\sin 2x + \cos 2x$ where $0 \leq \alpha \leq 360°$

$k\sin(2x + \alpha)° \;=\; k\sin 2x\cos\alpha + k\cos 2x\sin\alpha$

$\sqrt{3}\sin 2x + \cos 2x$

so $k\cos\alpha = \sqrt{3}$ and $k\sin\alpha = 1$

$k = \sqrt{a^2 + b^2}$

$k = \sqrt{\sqrt{3}^2 + 1^2}$

$k = \sqrt{3 + 1}$

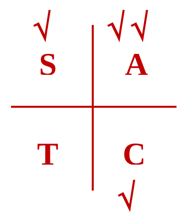
$k = \sqrt{4} = 2$

$Tan\alpha = \frac{k\sin\alpha}{k\cos\alpha}$

$Tan\alpha = \frac{1}{\sqrt{3}}$

$\alpha = \tan^{-1}(\frac{1}{\sqrt{3}})$

$\alpha = 30°$



Hence α is in 1st quadrant

So $\sqrt{3}\sin 2x + \cos 2x = 2\sin(2x + 30)°$

So Max = 2 and Min = –2

Maximum occurs when:

$2\sin(2x + 30)° = 2$

$\sin(2x + 30)° = 1$

$2x + 30° = \sin^{-1}(1)$

$2x + 30° = 90°$ or $450°$

$2x = 60°$ or $420°$

$x = 30°$ or $210°$

(30° , 2) & (210° , 2)

Minimum occurs when:

$2\sin(2x + 30)° = -2$

$\sin(2x + 30)° = -1$

$2x + 30° = \sin^{-1}(-1)$

$2x + 30° = 270°$ or $630°$

$2x = 240°$ or $600°$

$x = 120°$ or $300°$

(120° , –2) & (300° , –2)

Cuts y-axis when $x = 0$:

$y = 2\sin(2x + 30)°$

$y = 2\sin(2(0) + 30)°$

$y = 2\sin 30°$

$y = 2 \times 0.5$

$y = 1$ (0 , 1)

Cuts x-axis when $y = 0$:

$y = 2\sin(2x + 30)°$

$2\sin(2x + 30)° = 0$

$\sin(2x + 30)° = 0$

$2x + 30° = \sin^{-1}(0)$

$2x + 30° = 0, 180, 360, 540, 720$

$2x =$ ~~-30~~, 150, 330, 510, 690

$x = 75, 165, 255, 345$

The Graph of $y = 2\sin(2x + 30)°$